

# THE $m = 1$ AMPLITUHEDRON AND CYCLIC HYPERPLANE ARRANGEMENTS

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ABSTRACT. The *(tree) amplituhedron*  $\mathcal{A}_{n,k,m}$  is the image in the Grassmannian  $\text{Gr}_{k,k+m}$  of the totally nonnegative part of  $\text{Gr}_{k,n}$ , under a (map induced by a) linear map which is totally positive. It was introduced by Arkani-Hamed and Trnka in 2013 in order to give a geometric basis for the computation of scattering amplitudes in  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory. When  $k + m = n$ , the amplituhedron is isomorphic to the totally nonnegative Grassmannian, and when  $k = 1$ , the amplituhedron is a cyclic polytope. While the case  $m = 4$  is most relevant to physics, the amplituhedron is an interesting mathematical object for any  $m$ . In this paper we study it in the case  $m = 1$ . We start by taking an orthogonal point of view and define a related “B-amplituhedron”  $\mathcal{B}_{n,k,m}$ , which we show is isomorphic to  $\mathcal{A}_{n,k,m}$ . We use this reformulation to describe the amplituhedron in terms of sign variation. We then give a cell decomposition of the amplituhedron  $\mathcal{A}_{n,k,1}$  using the images of a collection of distinguished cells of the totally nonnegative Grassmannian. We also show that  $\mathcal{A}_{n,k,1}$  can be identified with the complex of bounded faces of a *cyclic hyperplane arrangement*, and describe how its cells fit together. We deduce that  $\mathcal{A}_{n,k,1}$  is homeomorphic to a ball.

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## 1. INTRODUCTION

The totally nonnegative Grassmannian  $\text{Gr}_{k,n}^{\geq 0}$  is the subset of the real Grassmannian  $\text{Gr}_{k,n}$  consisting of points with all Plücker coordinates nonnegative. Following seminal work of

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Lusztig [Lus94], as well as by Fomin and Zelevinsky [FZ99], Postnikov initiated the combinatorial study of  $\text{Gr}_{k,n}^{\geq 0}$  and its cell decomposition [Pos]. Since then the totally nonnegative Grassmannian has found applications in diverse contexts such as mirror symmetry [MR], soliton solutions to the KP equation [KW14], and scattering amplitudes for  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory [AHBC<sup>+</sup>16].

Building on [AHBC<sup>+</sup>16], Arkani-Hamed and Trnka [AHT14] recently introduced a beautiful new mathematical object called the *(tree) amplituhedron*, which is the image of the totally nonnegative Grassmannian under a particular map.

**Definition 1.1.** Let  $Z$  be a  $(k+m) \times n$  real matrix whose maximal minors are all positive, where  $m \geq 0$  is fixed with  $k+m \leq n$ . Then it induces a map

$$\tilde{Z} : \text{Gr}_{k,n}^{\geq 0} \rightarrow \text{Gr}_{k,k+m}$$

defined by

$$\tilde{Z}(\langle v_1, \dots, v_k \rangle) := \langle Z(v_1), \dots, Z(v_k) \rangle,$$

where  $\langle v_1, \dots, v_k \rangle$  is an element of  $\text{Gr}_{k,n}^{\geq 0}$  written as the span of  $k$  basis vectors.<sup>1</sup> The *(tree) amplituhedron*  $\mathcal{A}_{n,k,m}(Z)$  is defined to be the image  $\tilde{Z}(\text{Gr}_{k,n}^{\geq 0})$  inside  $\text{Gr}_{k,k+m}$ .

In special cases the amplituhedron recovers familiar objects. If  $Z$  is a square matrix, i.e.  $k+m = n$ , then  $\mathcal{A}_{n,k,m}(Z)$  is isomorphic to the totally nonnegative Grassmannian. If  $k = 1$ , then  $\mathcal{A}_{n,1,m}(Z)$  is a *cyclic polytope* in projective space [Stu88].

While the amplituhedron  $\mathcal{A}_{n,k,m}(Z)$  is an interesting mathematical object for any  $m$ , the case of immediate relevance to physics is  $m = 4$ . In this case, it provides a geometric basis for the computation of *scattering amplitudes* in  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory. These amplitudes are complex numbers related to the probability of observing a certain scattering process of  $n$  particles. It is expected that such amplitudes can be expressed (modulo higher-order terms) as an integral over the amplituhedron  $\mathcal{A}_{n,k,4}(Z)$ . This statement would follow from the conjecture of Arkani-Hamed and Trnka [AHT14] that the images of a certain collection of  $4k$ -dimensional cells of  $\text{Gr}_{k,n}^{\geq 0}$  provide a “triangulation” of the amplituhedron  $\mathcal{A}_{n,k,4}(Z)$ . More specifically, the BCFW recurrence [BCF05, BCFW05] provides one way to compute scattering amplitudes. Translated into the Grassmannian formulation of [AHBC<sup>+</sup>16], the terms in the BCFW recurrence can be identified with a collection of  $4k$ -dimensional cells in  $\text{Gr}_{k,n}^{\geq 0}$ . If the images of these *BCFW cells* in  $\mathcal{A}_{n,k,4}(Z)$  fit together in a nice way, then we can combine the contributions from each term into a single integral over  $\mathcal{A}_{n,k,4}(Z)$ .

In this paper, we study the amplituhedron  $\mathcal{A}_{n,k,1}(Z)$  for  $m = 1$ . We find that this object is already interesting and non-trivial. Since  $\mathcal{A}_{n,k,1}(Z) \subseteq \text{Gr}_{k,k+1}$ , it is convenient to take orthogonal complements and work with lines rather than  $k$ -planes in  $\mathbb{R}^{k+1}$ . This leads us to define a related “B-amplituhedron”

$$\mathcal{B}_{n,k,m}(W) := \{V^\perp \cap W : V \in \text{Gr}_{k,n}^{\geq 0}\} \subseteq \text{Gr}_m(W),$$

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<sup>1</sup>The fact that  $Z$  has positive maximal minors ensures that  $\tilde{Z}$  is well defined [AHT14]. See [Kar16, Theorem 4.2] for a characterization of when a matrix  $Z$  gives rise to a well-defined map  $\tilde{Z}$ .

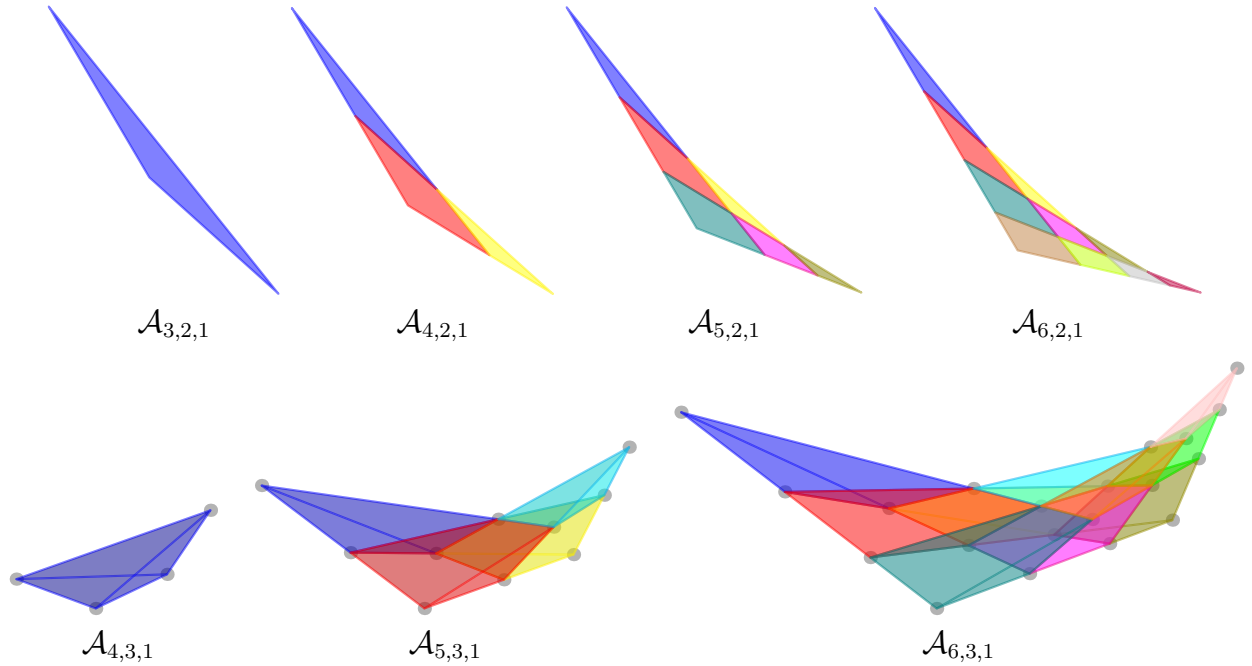


FIGURE 1. The amplituhedron  $\mathcal{A}_{n,k,1}(Z)$  as the complex of bounded faces of a cyclic hyperplane arrangement of  $n$  hyperplanes in  $\mathbb{R}^k$ , for  $k = 2, 3$  and  $n \leq 6$ .

which is homeomorphic to  $\mathcal{A}_{n,k,m}(Z)$ , where  $W$  is the subspace of  $\mathbb{R}^n$  spanned by the rows of  $Z$  (Section 3.1). In the context of scattering amplitudes ( $m = 4$ ),  $W$  is the span of 4 bosonic variables and  $k$  fermionic variables. Building on results of [Kar16], we use this reformulation to give a description of the amplituhedron  $\mathcal{A}_{n,k,m}(Z)$  in terms of sign variation (Section 3.3).

Modeling the  $m = 4$  case, we define a BCFW-like recursion in the case  $m = 1$ , which we use to produce a subset of  $k$ -dimensional “BCFW cells” of  $\text{Gr}_{k,n}^{\geq 0}$  (Section 4). The set of all cells of  $\text{Gr}_{k,n}^{\geq 0}$  are in bijection with various combinatorial objects, including J-diagrams and decorated permutations, so we describe our  $m = 1$  BCFW cells in terms of these objects. We then show that their images triangulate the  $m = 1$  amplituhedron; more specifically, we show that  $\mathcal{A}_{n,k,1}(Z)$  is homeomorphic to a  $k$ -dimensional subcomplex of the totally nonnegative Grassmannian  $\text{Gr}_{k,n}^{\geq 0}$  (Section 5). See Figure 8 for  $\mathcal{A}_{4,2,1}(Z)$  as a subcomplex of  $\text{Gr}_{2,4}^{\geq 0}$ .

We also show that  $\mathcal{A}_{n,k,1}(Z)$  can be identified with the complex of bounded faces of a certain hyperplane arrangement of  $n$  hyperplanes in  $\mathbb{R}^k$ , called a *cyclic hyperplane arrangement* (Section 6). We use this description of the  $m = 1$  amplituhedron to describe how its cells fit together (Section 7).

It is known that the totally nonnegative Grassmannian has a remarkably simple topology: it is contractible with boundary a sphere [RW10], and its poset of cells is Eulerian [Wil07]. While there are not yet any general results in this direction, calculations of Euler characteristics [FGMT15] indicate that the amplituhedron  $\mathcal{A}_{n,k,m}(Z)$  is likely also topologically

very nice. Our description of  $\mathcal{A}_{n,k,1}(Z)$  as the complex of bounded faces of a hyperplane arrangement, together with a result of Dong [Don08], implies that the  $m = 1$  amplituhedron is homeomorphic to a closed ball (Corollary 6.18).

Since  $k + m \leq n$ , the map

$$\tilde{Z} : \text{Gr}_{k,n}^{\geq 0} \rightarrow \text{Gr}_{k,k+m}$$

is far from injective in general. We determine when an arbitrary cell of  $\text{Gr}_{k,n}^{\geq 0}$  is mapped injectively by  $\tilde{Z}$  into  $\mathcal{A}_{n,k,1}(Z)$ , and in this case we describe its image in  $\mathcal{A}_{n,k,1}(Z)$  (Section 8).

Finally, we discuss to what extent our results hold in the setting of *Grassmann polytopes* (Section 9). Grassmann polytopes are generalizations of amplituhedra obtained by relaxing the positivity condition on the matrix  $Z$  [Lam16b].

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## 2. BACKGROUND ON THE TOTALLY NONNEGATIVE GRASSMANNIAN

The (*real*) *Grassmannian*  $\text{Gr}_{k,n}$  is the space of all  $k$ -dimensional subspaces of  $\mathbb{R}^n$ , for  $0 \leq k \leq n$ . An element of  $\text{Gr}_{k,n}$  can be viewed as a  $k \times n$  matrix of rank  $k$ , modulo left multiplications by nonsingular  $k \times k$  matrices. In other words, two  $k \times n$  matrices of rank  $k$  represent the same point in  $\text{Gr}_{k,n}$  if and only if they can be obtained from each other by row operations.

Let  $[n]$  denote  $\{1, \dots, n\}$ , and  $\binom{[n]}{k}$  the set of all  $k$ -element subsets of  $[n]$ . Given  $V \in \text{Gr}_{k,n}$  represented by a  $k \times n$  matrix  $A$ , for  $I \in \binom{[n]}{k}$  we let  $\Delta_I(V)$  be the maximal minor of  $A$  located in the column set  $I$ . The  $\Delta_I(V)$  do not depend on our choice of matrix  $A$  (up to simultaneous rescaling by a nonzero constant), and are called the *Plücker coordinates* of  $V$ .

**Definition 2.1** ([Pos, Section 3]). The *totally nonnegative Grassmannian*  $\text{Gr}_{k,n}^{\geq 0}$  is the set of elements  $V \in \text{Gr}_{k,n}$  such that  $\Delta_I(V) \geq 0$  for all  $I \in \binom{[n]}{k}$ . For  $M \subseteq \binom{[n]}{k}$ , the *positive Grassmann cell*  $S_M$  is the subset of elements  $V \in \text{Gr}_{k,n}^{\geq 0}$  with the prescribed collection of Plücker coordinates strictly positive (i.e.  $\Delta_I(V) > 0$  for all  $I \in M$ ), and the remaining Plücker coordinates equal to zero (i.e.  $\Delta_J(V) = 0$  for all  $J \in \binom{[n]}{k} \setminus M$ ). We call  $M$  a *positroid* if  $S_M$  is nonempty.

**Remark 2.2.** There is an action of the “positive torus”  $T_{>0} = \mathbb{R}_{>0}^n$  on  $\text{Gr}_{k,n}^{\geq 0}$ . Concretely, if  $A$  is a  $k \times n$  matrix representing an element of  $\text{Gr}_{k,n}^{\geq 0}$ , then the positive torus acts on  $A$  by rescaling its columns. If  $\mathbf{t} = (t_1, \dots, t_n) \in T_{>0}$  and  $A$  represents an element of  $S_M$ , then  $\mathbf{t} \cdot A$  also represents an element of  $S_M$ .

**Definition 2.3.** We let  $Q_{k,n}$  denote the poset on cells defined by  $S_M \leq S_{M'}$  if and only if  $S_M \subseteq \overline{S_{M'}}$ .

The fact that each nonempty  $S_M$  is a topological cell is due to Postnikov [Pos]. Moreover, it was shown in [PSW09] that the cells glue together to form a CW decomposition of  $\text{Gr}_{k,n}^{\geq 0}$ .

**2.1. Combinatorial objects parameterizing cells.** In [Pos], Postnikov gave several families of combinatorial objects in bijection with cells of the totally nonnegative Grassmannian. In this section we will start by defining  $\mathcal{J}$ -diagrams, decorated permutations, and equivalence classes of reduced plabic graphs, and give (compatible) bijections among all these objects. This will give us a canonical way to label each positroid by a  $\mathcal{J}$ -diagram, a decorated permutation, and an equivalence class of plabic graphs.

**Definition 2.4.** A decorated permutation of the set  $[n]$  is a bijection  $\pi : [n] \rightarrow [n]$  whose fixed points are colored either black or white. We denote a black fixed point by  $\pi(i) = \underline{i}$  and a white fixed point by  $\pi(i) = \bar{i}$ . An anti-excedance of the decorated permutation  $\pi$  is an element  $i \in [n]$  such that either  $\pi^{-1}(i) > i$  or  $\pi(i) = \bar{i}$  (i.e.  $i$  is a white fixed point).

**Definition 2.5.** Fix  $k$  and  $n$ . Given a partition  $\lambda$ , we let  $Y_\lambda$  denote the Young diagram associated to  $\lambda$ . A  $\mathcal{J}$ -diagram (or Le-diagram)  $D$  of shape  $\lambda$  and type  $(k, n)$  is a Young diagram of shape  $Y_\lambda$  contained in a  $k \times (n - k)$  rectangle, whose boxes are filled with 0's and +'s in such a way that the  $\mathcal{J}$ -property is satisfied: there is no 0 which has a + above it in the same column and a + to its left in the same row. See Figure 2 for an example of a  $\mathcal{J}$ -diagram.

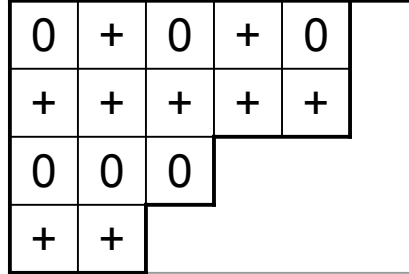


FIGURE 2. A  $\mathcal{J}$ -diagram with  $\lambda = (5, 5, 3, 2)$ ,  $n = 10$ , and  $k = 4$ .

**Lemma 2.6** ([Pos, Section 20]). *The following algorithm is a bijection between  $\mathcal{J}$ -diagrams  $D$  of type  $(k, n)$  and decorated permutations  $\pi$  on  $[n]$  with exactly  $k$  anti-excedances.*

- (1) Replace each + in the  $\mathcal{J}$ -diagram  $D$  with an elbow joint  $\searrow$ , and each 0 in  $D$  with a cross  $\begin{smallmatrix} \text{---} \\ | \\ \text{---} \end{smallmatrix}$ .
- (2) Note that the southeast border of  $Y_\lambda$  gives rise to a length- $n$  path from the northeast corner to the southwest corner of the  $k \times (n - k)$  rectangle. Label the edges of this path with the numbers 1 through  $n$ .
- (3) Now label the edges of the north and west border of  $Y_\lambda$  so that opposite horizontal edges and opposite vertical edges have the same label.
- (4) View the resulting ‘pipe dream’ as a permutation  $\pi = \pi(D)$  on  $[n]$ , by following the ‘pipes’ from the southeaster border to the northwest border of the Young diagram. If the pipe originating at label  $i$  ends at the label  $j$ , we define  $\pi(i) := j$ .

- (5) If  $\pi(i) = i$  and  $i$  labels two horizontal (respectively, vertical) edges of  $Y_\lambda$ , then  $\pi(i) := \underline{i}$  (respectively,  $\pi(i) := \bar{i}$ ).

Figure 3 illustrates this procedure.

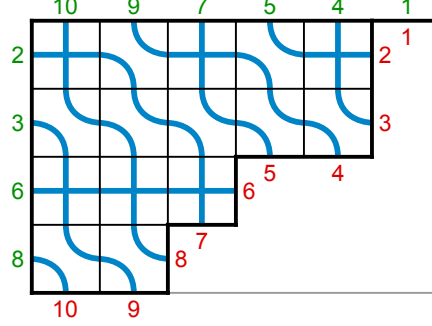


FIGURE 3. The pipe dream corresponding to the J-diagram in Figure 2, with decorated permutation  $\pi = (\underline{1}, 5, 4, 9, 7, \bar{6}, 2, 10, 3, 8)$ .

**Definition 2.7.** A *plabic graph*<sup>2</sup> is an undirected planar graph  $G$  drawn inside a disk (considered modulo homotopy) with  $n$  boundary vertices on the boundary of the disk, labeled  $1, \dots, n$  in clockwise order, as well as some colored *internal vertices*. These internal vertices are strictly inside the disk and are each colored either black or white. Moreover, each boundary vertex  $i$  in  $G$  is incident to a single edge. If a boundary vertex is adjacent to a leaf (a vertex of degree 1), we refer to that leaf as a *lollipop*.

A *perfect orientation*  $\mathcal{O}$  of a plabic graph  $G$  is a choice of orientation of each of its edges such that each black internal vertex  $u$  is incident to exactly one edge directed away from  $u$ , and each white internal vertex  $v$  is incident to exactly one edge directed towards  $v$ . A plabic graph is called *perfectly orientable* if it admits a perfect orientation. Let  $G_{\mathcal{O}}$  denote the directed graph associated with a perfect orientation  $\mathcal{O}$  of  $G$ . Since each boundary vertex is incident to a single edge, it is either a *source* (if it is incident to an outgoing edge) or a *sink* (if it is incident to an incoming edge) in  $G_{\mathcal{O}}$ . The *source set*  $I_{\mathcal{O}} \subset [n]$  is the set of boundary vertices which are sources in  $G_{\mathcal{O}}$ .

Figure 5 shows a plabic graph with a perfect orientation. In that example,  $I_{\mathcal{O}} = \{2, 3, 6, 8\}$ .

All perfect orientations of a fixed plabic graph  $G$  have source sets of the same size  $k$ , where  $k - (n - k) = \sum \text{color}(v) \cdot (\deg(v) - 2)$ . Here the sum is over all internal vertices  $v$ , where  $\text{color}(v) = 1$  if  $v$  is black, and  $\text{color}(v) = -1$  if  $v$  is white; see [Pos, Lemma 9.4]. In this case we say that  $G$  is of *type*  $(k, n)$ .

The following construction of Postnikov [Pos, Sections 6 and 20] associates a perfectly orientable plabic graph to any J-diagram.

**Definition 2.8.** Let  $D$  be a J-diagram and  $\pi$  its decorated permutation. Delete the 0's of  $D$ , and replace each  $+$  with a vertex. From each vertex we construct a hook which goes east and south, to the border of the Young diagram. The resulting diagram is called the

<sup>2</sup>“Plabic” stands for *planar bi-colored*.

hook diagram  $H(D)$ . After replacing the edges along the southeast border of the Young diagram with boundary vertices labeled by  $1, \dots, n$ , we obtain a planar graph in a disk, with  $n$  boundary vertices and one internal vertex for each  $+$  of  $D$ . Then we replace the local region around each internal vertex as in Figure 4, and add a black (respectively, white) lollipop for each black (respectively, white) fixed point of  $\pi$ . This gives rise to a plabic graph which we call  $G(D)$ . By orienting the edges of  $G(D)$  down and to the left, we obtain a perfect orientation.

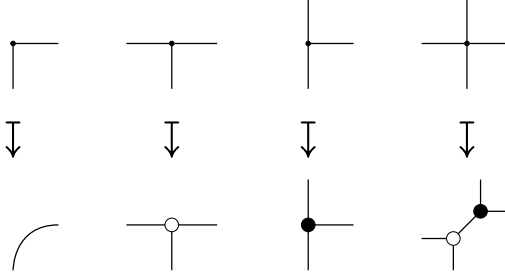


FIGURE 4. Local substitutions for getting the plabic graph  $G(D)$  from the hook diagram  $H(D)$ .

Figure 5a depicts the hook diagram  $H(D)$  corresponding to the J-diagram  $D$  from Figure 2, and Figure 5b shows the corresponding plabic graph  $G(D)$ .

More generally, each J-diagram  $D$  is associated with a family of *reduced plabic graphs* consisting of  $G(D)$  together with other plabic graphs which can be obtained from  $G(D)$  by certain *moves*; see [Pos, Section 12].

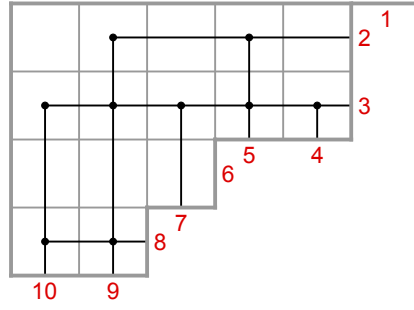
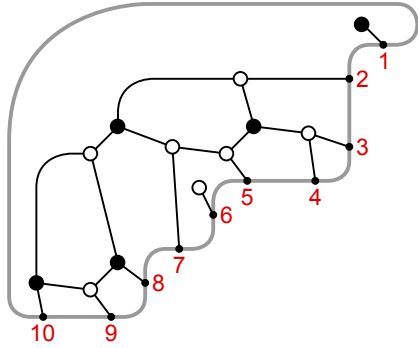
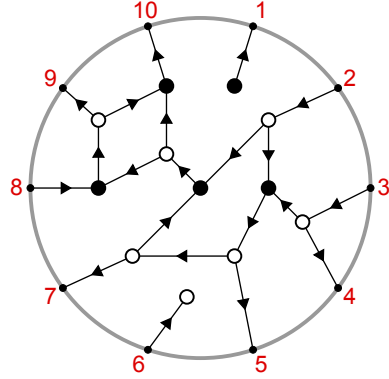
From the plabic graph constructed in Definition 2.8 (and more generally from a reduced plabic graph  $G$ ), one may read off the corresponding decorated permutation  $\pi_G$  as follows.

**Definition 2.9.** Let  $G$  be a reduced plabic graph as above with boundary vertices  $1, \dots, n$ . The *trip from  $i$*  is the path obtained by starting from  $i$  and traveling along edges of  $G$  according to the rule that each time we reach an internal black vertex we turn (maximally) right, and each time we reach an internal white vertex we turn (maximally) left. This trip ends at some boundary vertex  $\pi(i)$ . By [Pos, Section 13], the fact that  $G$  is reduced implies that each fixed point of  $\pi$  is attached to a lollipop; we color each fixed point by the color of its lollipop. In this way we obtain the *decorated permutation*  $\pi_G = (\pi(1), \dots, \pi(n))$  of  $G$ .

We invite the reader to verify that when we apply these rules to plabic graph  $G$  of Figure 5b, we obtain the decorated permutation  $\pi_G = (\underline{1}, 5, 4, 9, 7, \bar{6}, 2, 10, 3, 8)$ .

**2.2. Matroids and positroids.** A matroid is a combinatorial object which unifies several notions of independence. Among the many equivalent ways of defining a matroid we will adopt the point of view of bases, which is one of the most convenient for the study of positroids. We refer the reader to [Oxl11] for an in-depth introduction to matroid theory.

**Definition 2.10.** Let  $E$  be a finite set. A *matroid with ground set  $E$*  is a subset  $M \subseteq 2^E$  satisfying the *basis exchange axiom*:

(A) The hook diagram  $H(D)$ .(B) The plabic graph  $G(D)$ .(C) The plabic graph  $G(D)$  redrawn and perfectly oriented.FIGURE 5. The hook diagram and plabic graph associated to the  $\mathcal{I}$ -diagram  $D$  from Figure 2.

if  $B, B' \in M$  and  $b \in B \setminus B'$ , then there exists  $b' \in B' \setminus B$  such that  $(B \setminus \{b\}) \cup \{b'\} \in M$ . The elements of  $M$  are called *bases*. All bases of  $M$  have the same size, called the *rank* of  $M$ . We say that  $i \in E$  is a *loop* if  $i$  is contained in no basis of  $M$ , and a *coloop* if  $i$  is contained in every basis of  $M$ .

**Example 2.11.** Let  $A$  be a  $k \times n$  matrix of rank  $k$  with entries in a field  $\mathbb{F}$ . Then the subsets  $B \in \binom{[n]}{k}$  such that the columns  $B$  of  $A$  are linearly independent form the bases of a matroid  $M(A)$  with rank  $k$  and ground set  $[n]$ . In terms of the Grassmannian, the rows of  $A$  span an element of  $\text{Gr}_{k,n}(\mathbb{F})$ , whose nonzero Plücker coordinates are indexed by  $M(A)$ . For example, the matrix

$$A := \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 \end{bmatrix}$$

over  $\mathbb{F} := \mathbb{Q}$  gives rise to the matroid  $M(A) = \{\{1, 2\}, \{1, 3\}, \{1, 5\}, \{2, 3\}, \{3, 5\}\}$  with ground set  $[5]$ . In this example, 4 is a loop of  $M(A)$ , and  $M(A)$  has no coloops.

Matroids arising in this way are called *representable* (over  $\mathbb{F}$ ).  $\diamond$



**Example 2.12.** Given  $k \in \mathbb{N}$  and a finite set  $E$ , all  $k$ -subsets of  $E$  form a matroid  $M$ , called the *uniform matroid* (of rank  $k$  with ground set  $E$ ). Note that  $M$  can be represented over any infinite field, by a generic  $k \times n$  matrix.  $\diamond$

Recall the definition of a positroid from Definition 2.1. In the language of matroid theory, a positroid is a matroid representable by an element of the totally nonnegative Grassmannian. Every perfectly orientable plabic graph gives rise to a positroid as follows.

**Definition 2.13** ([Pos, Proposition 11.7]). Let  $G$  be a plabic graph of type  $(k, n)$ . Then we have a positroid  $M_G$  on  $[n]$  whose bases are precisely

$$\{I_{\mathcal{O}} : \mathcal{O} \text{ is a perfect orientation of } G\},$$

where  $I_{\mathcal{O}}$  is the set of sources of  $\mathcal{O}$ .

If  $D$  is a  $\mathbb{J}$ -diagram contained in a  $k \times (n - k)$  rectangle, we let  $M(D)$  denote the positroid  $M_{G(D)}$  of the plabic graph  $G(D)$  from Definition 2.8.

Postnikov [Pos, Theorem 17.1] showed that every positroid can be realized as  $M(D)$  for some  $\mathbb{J}$ -diagram  $D$ . We observe that we can describe the loops and coloops of  $M(D)$  in terms of  $D$  as follows:  $i$  is a loop if and only if  $i$  labels a horizontal step whose column contains only 0's, and  $i$  is a coloop if and only if  $i$  labels a vertical step in the southeast border whose row contains only 0's.

We introduce some further notions from matroid theory which we will use later: duality, direct sum, connectedness, restriction, and a partial order.

**Definition 2.14.** Let  $M$  be a matroid with ground set  $E$ . Then  $\{E \setminus B : B \in M\}$  is the set of bases of a matroid  $M^*$  with ground set  $E$ , called the *dual* of  $M$ .

See [Oxl11, Section 2] for a proof that  $M^*$  is indeed a matroid. We make the following observations about matroid duality:

- $(M^*)^* = M$ ;
- the ranks of  $M$  and  $M^*$  sum to  $|E|$ ;
- $i \in E$  is a loop of  $M$  if and only if  $i$  is a coloop of  $M^*$ ;
- if  $E = [n]$ , then  $M$  is a positroid if and only if  $M^*$  is a positroid (see Lemma 3.3(ii)).

**Example 2.15.** Let  $A := \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 \end{bmatrix}$ , as in Example 2.11. Then

$$M(A)^* = \{\{3, 4, 5\}, \{2, 4, 5\}, \{2, 3, 4\}, \{1, 4, 5\}, \{1, 2, 4\}\},$$

and is represented by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ -2 & 0 & 1 & 0 & -3 \end{bmatrix},$$

whose rows are orthogonal to the rows of  $A$ .  $\diamond$

**Definition 2.16.** Let  $M$  and  $N$  be matroids with ground sets  $E$  and  $F$ , respectively. The *direct sum*  $M \oplus N$  is the matroid with ground set  $E \sqcup F$ , and bases  $\{B \sqcup C : B \in M, C \in N\}$ . The rank of  $M \oplus N$  is the sum of the ranks of  $M$  and  $N$ .

A matroid is *connected* if we cannot write it as the direct sum of two matroids whose ground sets are nonempty. Any matroid  $M$  can be written uniquely (up to permuting the summands) as the direct sum of connected matroids, whose ground sets are the *connected components* of  $M$ ; see [Oxl11, Corollary 4.2.9].

**Example 2.17.** Consider the matrix  $A := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$  and its associated matroid  $M(A) = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\}$ . We have  $M(A) = M_1 \oplus M_2$ , where  $M_1$  is the uniform matroid of rank 1 with ground set  $\{1, 4\}$ , and  $M_2$  is the uniform matroid of rank 1 with ground set  $\{2, 3\}$ . Since  $M_1$  and  $M_2$  are connected, the connected components of  $M(A)$  are  $\{1, 4\}$  and  $\{2, 3\}$ . In particular,  $M(A)$  is disconnected.  $\diamond$

**Definition 2.18.** Let  $M$  be a matroid with ground set  $E$ . For a subset  $F \subseteq E$ , the *restriction*  $M|_F$  of  $M$  to  $F$  is the matroid with ground set  $F$  whose bases are the inclusion-maximal sets of  $\{B \cap F : B \in M\}$ ; see [Oxl11, p. 20].

For example, if  $M$  and  $N$  are matroids with ground sets  $E$  and  $F$ , respectively, then the restriction of  $M \oplus N$  to  $E$  is  $M$ , and the restriction of  $M \oplus N$  to  $F$  is  $N$ .

**Definition 2.19.** We define a partial order on matroids of rank  $k$  with ground set  $E$  as follows:  $M' \leq M$  if and only if every basis of  $M'$  is a basis of  $M$ . (In the matroid theory literature, one says that the identity map on  $E$  is a *weak map* from  $M$  to  $M'$ .)

Note that  $M' \leq M$  if and only if  $M'^* \leq M^*$ . This partial order, restricted to positroids of rank  $k$  with ground set  $[n]$ , recovers the poset  $Q_{k,n}$  of cells of  $\text{Gr}_{k,n}^{\geq 0}$  coming from containment of closures [Pos, Section 17]. The poset  $Q_{2,4}$  for  $\text{Gr}_{2,4}^{\geq 0}$  is shown in Figure 8.

**Remark 2.20.** All bijections that we have defined in this section are compatible. This gives us a canonical way to label each positroid of rank  $k$  with ground set  $[n]$  by a set of bases, a decorated permutation, a J-diagram, and an equivalence class of reduced plabic graphs. The partial order on positroids (Definition 2.19) gives a partial order on these other objects (of type  $(k, n)$ ).

### 3. A COMPLEMENTARY VIEW OF THE AMPLITUHEDRON $\mathcal{A}_{n,k,1}$

#### 3.1. Background on sign variation.

**Definition 3.1.** Given  $v \in \mathbb{R}^n$ , let  $\text{var}(v)$  be the number of times  $v$  changes sign, when viewed as a sequence of  $n$  numbers and ignoring any zeros. We use the convention  $\text{var}(0) := -1$ . We also define

$$\overline{\text{var}}(v) := \max\{\text{var}(w) : w \in \mathbb{R}^n \text{ such that } w_i = v_i \text{ for all } i \in [n] \text{ with } v_i \neq 0\},$$

i.e.  $\overline{\text{var}}(v)$  is the maximum number of times  $v$  changes sign after we choose a sign for each zero component.

For example, if  $v := (4, -1, 0, -2) \in \mathbb{R}^4$ , then  $\text{var}(v) = 1$  and  $\overline{\text{var}}(v) = 3$ .

We now explain how  $\text{var}(\cdot)$  and  $\overline{\text{var}}(\cdot)$  are dual to each other.

**Definition 3.2.** We define  $\text{alt} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $\text{alt}(v) := (v_1, -v_2, v_3, -v_4, \dots, (-1)^{n-1}v_n)$  for  $v \in \mathbb{R}^n$ . If  $S \subseteq \mathbb{R}^n$ , we let  $\text{alt}(S)$  denote  $\{\text{alt}(v) : v \in S\}$ .

**Lemma 3.3** (Duality via  $\text{alt}$ ).

- (i)<sup>3</sup> [GK50] We have  $\text{var}(v) + \overline{\text{var}}(\text{alt}(v)) = n - 1$  for all  $v \in \mathbb{R}^n \setminus \{0\}$ .
- (ii)<sup>4</sup> [Hil90, Hoc75] Given  $V \in \text{Gr}_{k,n}$ , let  $V^\perp \in \text{Gr}_{n-k,n}$  be the orthogonal complement of  $V$ . Then  $V$  and  $\text{alt}(V^\perp)$  have the same Plücker coordinates, i.e.  $\Delta_I(V) = \Delta_{[n] \setminus I}(\text{alt}(V^\perp))$  for all  $I \in \binom{[n]}{k}$ .

Note that part (ii) above implies that a subspace  $V$  is totally nonnegative if and only if  $\text{alt}(V^\perp)$  is totally nonnegative, and totally positive if and only if  $\text{alt}(V^\perp)$  is totally positive.

The following result of Gantmakher and Krein, which characterizes totally nonnegative and totally positive subspaces in terms of sign variation, will be essential for us.

**Theorem 3.4** ([GK50, Theorems V.3, V.7, V.1, V.6]). Let  $V \in \text{Gr}_{k,n}$ .

- (i)  $V \in \text{Gr}_{k,n}^{\geq 0} \iff \text{var}(v) \leq k - 1$  for all  $v \in V \iff \overline{\text{var}}(w) \geq k$  for all  $w \in V^\perp \setminus \{0\}$ .
- (ii)  $V \in \text{Gr}_{k,n}^{> 0} \iff \overline{\text{var}}(v) \leq k - 1$  for all  $v \in V \setminus \{0\} \iff \text{var}(w) \geq k$  for all  $w \in V^\perp \setminus \{0\}$ .

**Corollary 3.5.** If  $V \in \text{Gr}_{k,n}^{\geq 0}$  and  $W \in \text{Gr}_{r,n}^{> 0}$ , where  $r \geq k$ , then  $V \cap W^\perp = \{0\}$ .

**Proof.** By Theorem 3.4,  $v \in V$  implies that  $\text{var}(v) \leq k - 1$ . And  $w \in W^\perp \setminus \{0\}$  implies that  $\text{var}(w) \geq r \geq k$ . Therefore  $V \cap W^\perp = \{0\}$ . ■

We will also need to know which sign vectors appear in elements of  $\text{Gr}_{k,n}^{> 0}$ .

**Definition 3.6.** For  $t \in \mathbb{R}$  we define

$$\text{sign}(t) := \begin{cases} 0, & \text{if } t = 0 \\ +, & \text{if } t > 0 \\ -, & \text{if } t < 0 \end{cases}.$$

(We will sometimes use 1 and  $-1$  in place of  $+$  and  $-$ .) Given  $v \in \mathbb{R}^n$ , define the *sign vector*  $\text{sign}(v) \in \{0, +, -\}^n$  of  $v$  by  $\text{sign}(v)_i := \text{sign}(v_i)$  for  $i \in [n]$ . For example,  $\text{sign}(5, 0, -1, 2) = (+, 0, -, +) = (1, 0, -1, 1)$ . If  $S \subseteq \mathbb{R}^n$ , we let  $\text{sign}(S)$  denote  $\{\text{sign}(v) : v \in S\}$ .

**Lemma 3.7.**

Suppose that  $V \in \text{Gr}_{k,n}^{> 0}$  with orthogonal complement  $V^\perp$ .

- (i)  $\text{sign}(V) = \{\sigma \in \{0, +, -\}^n : \overline{\text{var}}(\sigma) \leq k - 1\} \cup \{0\}$ .
- (ii)  $\text{sign}(V^\perp) = \{\sigma \in \{0, +, -\}^n : \text{var}(\sigma) \geq k\} \cup \{0\}$ .

This result essentially follows from Theorem 3.4, [BLVS<sup>+</sup>99, Proposition 9.4.1], and Lemma 3.3. For a more thorough explanation, see the claim in the proof of [Kar16, Lemma 4.1].

<sup>3</sup>This result is stated without proof as [GK50, Equation II.(67)]. See [And87, Equation (5.1)] for a proof.

<sup>4</sup>The earliest reference we found for this result is [Hoc75, Section 7], where it appears without proof. Hochster says this result “was basically known to Hilbert.” The idea is that if  $[I_k|A]$  is a  $k \times n$  matrix whose rows span  $V \in \text{Gr}_{k,n}$ , where  $A$  is a  $k \times (n - k)$  matrix, then  $V^\perp$  is the row span of the matrix  $[A^T | -I_{n-k}]$ . This appears implicitly in [Hil90, Equation (14)], and more explicitly in [Oxl11, Theorem 2.2.8] and [MR14, Proposition 3.1(i)].

**3.2. An orthogonally complementary view of the amplituhedron  $\mathcal{A}_{n,k,m}(Z)$ .** The amplituhedron  $\mathcal{A}_{n,k,m}(Z)$  is a subset of  $\text{Gr}_{k,k+m}$ . Since we are considering the case  $m = 1$ , it will be convenient for us to take orthogonal complements and work with subspaces of dimension  $m$ , rather than codimension  $m$ . To this end, we define an object  $\mathcal{B}_{n,k,m}(W) \subseteq \text{Gr}_m(W)$  for  $W \in \text{Gr}_{k+m,n}^{>0}$ , which we show is homeomorphic to  $\mathcal{A}_{n,k,m}(Z)$  (Proposition 3.12), where  $W = \text{rowspan}(Z)$ . We remark that in the context of scattering amplitudes when  $m = 4$ ,  $W$  is the subspace of  $\mathbb{R}^n$  spanned by 4 bosonic variables and  $k$  fermionic variables.

**Definition 3.8.** Given  $W \in \text{Gr}_{k+m,n}^{>0}$ , let

$$\mathcal{B}_{n,k,m}(W) := \{V^\perp \cap W : V \in \text{Gr}_{k,n}^{\geq 0}\} \subseteq \text{Gr}_m(W),$$

where  $\text{Gr}_m(W)$  denotes the subset of  $\text{Gr}_{m,n}$  of elements  $X \in \text{Gr}_{m,n}$  with  $X \subseteq W$ . Let us show that  $\mathcal{B}_{n,k,m}(W)$  is well defined, i.e.  $\dim(V^\perp \cap W) = m$  for all  $V \in \text{Gr}_{k,n}^{\geq 0}$ . By Corollary 3.5 we have  $V \cap W^\perp = \{0\}$ , so the sum  $V + W^\perp$  is direct. Hence  $\dim(V + W^\perp) = \dim(V) + \dim(W^\perp) = n - m$ . Since  $(V^\perp \cap W)^\perp = V + W^\perp$ , we get  $\dim(V^\perp \cap W) = m$ . (We remark that this is the same idea used in [Kar16, Section 4] to determine when, given an arbitrary linear map  $Z : \mathbb{R}^n \rightarrow \mathbb{R}^{k+m}$ , the image  $\tilde{Z}(\text{Gr}_{k,n}^{\geq 0})$  is well defined in  $\text{Gr}_{k,k+m}$ .)

**Remark 3.9.** While we were preparing this manuscript, we noticed that a similar construction appeared in Lam’s definition of *universal amplituhedron varieties* [Lam16b, Section 18]. There are two main differences between his construction and ours. First, Lam allows  $Z$  to vary (hence the term “universal”). Second, he works with complex varieties, and does not impose any positivity conditions on  $V$  or  $Z$  (rather, he restricts  $V$  to lie in a closed complex positroid cell in  $\text{Gr}_{k,n}(\mathbb{C})$ ). Correspondingly he works with rational maps, while we will need our maps to be well defined everywhere.

We now show that  $\mathcal{B}_{n,k,m}(W)$  is homeomorphic to  $\mathcal{A}_{n,k,1}(Z)$ , where  $Z$  is any  $(k+m) \times n$  matrix ( $n \geq k+m$ ) with positive maximal minors and row span  $W$ . The idea is that we obtain  $\mathcal{B}_{n,k,m}(W)$  from  $\mathcal{A}_{n,k,1}(Z) \subseteq \text{Gr}_{k,k+m}$  by taking orthogonal complements in  $\mathbb{R}^{k+m}$ , and then applying an isomorphism from  $\mathbb{R}^{k+m}$  to  $W$ , so that our subspaces lie in  $W$ , not  $\mathbb{R}^{k+m}$ .

**Lemma 3.10.** Let  $Z : \mathbb{R}^n \rightarrow \mathbb{R}^{k+m}$  be a surjective linear map, which we also regard as a  $(k+m) \times n$  matrix, and let  $W \in \text{Gr}_{k+m,n}$  be the row span of  $Z$ . Then the map  $f_Z : \text{Gr}_m(W) \rightarrow \text{Gr}_{k,k+m}$  given by

$$f_Z(X) := Z(X^\perp) = \{Z(x) : x \in X^\perp\} \quad \text{for all } X \in \text{Gr}_m(W)$$

is well defined and an isomorphism. (Here  $X^\perp \in \text{Gr}_{n-m,n}$  denotes the orthogonal complement of  $X$  in  $\mathbb{R}^n$ ; we use the notation  $Z(X^\perp)$ , and not  $\tilde{Z}(X^\perp)$ , because  $\dim(X^\perp) \neq k$ .)

Moreover, for  $X \in \text{Gr}_m(W)$  with corresponding point  $Y := f_Z(X) \in \text{Gr}_{k,k+m}$ , we can write the Plücker coordinates of  $X$  (as an element of  $\text{Gr}_{m,n}$ ) in terms of  $Y$  and  $Z$ , as follows. Let  $z_1, \dots, z_n \in \mathbb{R}^{k+m}$  be the columns of the  $(k+m) \times n$  matrix  $Z$ , and  $y_1, \dots, y_k \in \mathbb{R}^{k+m}$  be a basis of  $Y$ . Then for  $1 \leq i_1 < \dots < i_m \leq n$ , we have

$$(3.11) \quad \Delta_{\{i_1, \dots, i_m\}}(X) = \det([y_1 \mid \dots \mid y_k \mid z_{i_1} \mid \dots \mid z_{i_m}]).$$

The formula (3.11) is stated in [Lam16b, Section 18], though the proof is deferred to a forthcoming paper.

**Proof.** First let us show that  $f_Z$  is well defined, i.e. for  $X \in \text{Gr}_m(W)$ , we have  $\dim(f_Z(X)) = k$ . Since  $X \subseteq W$ , we have  $W^\perp \subseteq X^\perp$ , so we can write  $X^\perp = V \oplus W^\perp$  for some  $V \in \text{Gr}_{k,n}$ . Since  $\ker(Z) = W^\perp$ , we have  $Z(X^\perp) = Z(V + W^\perp) = Z(V)$ , and then  $V \cap \ker(Z) = V \cap W^\perp = \{0\}$  implies  $\dim(Z(V)) = k$ .

To see that  $f_Z$  is injective, suppose that we have  $X, X' \in \text{Gr}_m(W)$  with  $Z(X^\perp) = Z(X'^\perp)$ . Then as above we have  $X^\perp = V \oplus W^\perp$  and  $(X')^\perp = V' \oplus W^\perp$  where  $V \cap \ker(Z) = \{0\} = V' \cap \ker(Z)$ . But then  $Z(X^\perp) = Z(X'^\perp)$  implies that  $Z(V) = Z(V')$ , which implies that  $V = V'$  and hence  $X = X'$ . Now we describe the inverse of  $f_Z$ . Given  $Y \in \text{Gr}_{k,k+m}$ , consider the subspace  $Z^{-1}(Y) = \{v \in \mathbb{R}^n : Z(v) \in Y\}$ . Since  $Z^{-1}(Y)$  contains  $\ker(Z) = W^\perp$ , which has dimension  $n - k - m$ , and  $\dim(Y) = k$ , we have  $\dim(Z^{-1}(Y)) = n - m$ . Therefore we can write  $Z^{-1}(Y) = X^\perp$  for some  $X \in \text{Gr}_m(W)$ , and then  $Y = f_Z(X)$ . It follows that  $f_Z$  is invertible, and hence an isomorphism.

Now given  $X \in \text{Gr}_m(W)$  and  $Y := f_Z(X) \in \text{Gr}_{k,k+m}$ , we prove (3.11). Fix column vectors  $y_1, \dots, y_k \in \mathbb{R}^{k+m}$  which form a basis of  $Y$  and  $x_1, \dots, x_{n-m} \in \mathbb{R}^n$  which form a basis of  $X^\perp$ . For  $i \in [n-m]$ , we can write  $Z(x_i) = \sum_{j=1}^k C_{i,j} y_j$  for some  $C_{i,j} \in \mathbb{R}$ . Let  $D$  be the  $(n-m) \times n$  matrix whose rows are  $x_1^T, \dots, x_{n-m}^T$ . Then the elements of  $\text{Gr}_{n-m,n+k}$  and  $\text{Gr}_{k+m,n+k}$  which are given by the row spans of

$$[-C \mid D] \quad \text{and} \quad [y_1 \mid \dots \mid y_k \mid Z],$$

respectively, are orthogonally complementary. Given  $I = \{i_1 < \dots < i_m\} \subseteq [n]$  with complement  $[n] \setminus I = \{j_1 < \dots < j_{n-m}\}$ , applying Lemma 3.3(ii) twice gives

$$\begin{aligned} \Delta_I(X) &= \Delta_{[n] \setminus I}(\text{alt}(X^\perp)) = \Delta_{\{k+j_1, \dots, k+j_{n-m}\}}(\text{alt}([-C \mid D])) = \\ &= \Delta_{\{1, \dots, k, k+i_1, \dots, k+i_m\}}([y_1 \mid \dots \mid y_k \mid Z]) = \det([y_1 \mid \dots \mid y_k \mid z_{i_1} \mid \dots \mid z_{i_m}]). \quad \blacksquare \end{aligned}$$

**Proposition 3.12.** *Suppose that  $Z$  is a  $(k+m) \times n$  matrix ( $n \geq k+m$ ) with positive maximal minors, and  $W \in \text{Gr}_{k+m,n}^{>0}$  is the row span of  $Z$ . Then the map  $f_Z$  from Lemma 3.10 restricts to a homeomorphism from  $\mathcal{B}_{n,k,m}(W)$  onto  $\mathcal{A}_{n,k,m}(Z)$ , which sends  $V^\perp \cap W$  to  $\tilde{Z}(V)$  for all  $V \in \text{Gr}_{k,n}^{>0}$ . The Plücker coordinates of  $V^\perp \cap W$  can be written in terms of  $\tilde{Z}(V)$  and  $Z$  by (3.11).*

**Example 3.13.** Let  $(n, k, m) := (4, 2, 1)$ , and  $Z : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  be given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} =: [z_1 \mid z_2 \mid z_3 \mid z_4],$$

whose  $3 \times 3$  minors are all positive. Also let  $V \in \text{Gr}_{2,4}^{>0}$  be the row span of the matrix  $\begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & b \end{bmatrix}$ , where  $a, b \geq 0$ , and define  $Y := \tilde{Z}(V)$ . We can explicitly find a basis  $y_1, y_2 \in \mathbb{R}^3$  of  $Y$  as follows:

$$[y_1 \mid y_2] := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & 0 \\ 0 & 1 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & b \\ a & -b \\ 0 & 1+b \end{bmatrix}.$$

Now let  $X := V^\perp \cap \text{rowspan}(Z)$ , so that  $X \in \mathcal{B}_{4,2,1}(\text{rowspan}(Z))$  is mapped to  $Y \in \mathcal{A}_{4,2,1}(Z)$  under the homeomorphism of Proposition 3.12. We can write  $X$  as the line spanned by  $(a(b+1), -(b+1), -b(a+1), a+1)$ . We can check that we have

$$\begin{aligned} \Delta_{\{1\}}(X) &= \begin{vmatrix} 1 & b & 1 \\ a & -b & 0 \\ 0 & 1+b & 0 \end{vmatrix}, & \Delta_{\{3\}}(X) &= \begin{vmatrix} 1 & b & 0 \\ a & -b & 0 \\ 0 & 1+b & 1 \end{vmatrix}, \\ \Delta_{\{2\}}(X) &= \begin{vmatrix} 1 & b & 0 \\ a & -b & 1 \\ 0 & 1+b & 0 \end{vmatrix}, & \Delta_{\{4\}}(X) &= \begin{vmatrix} 1 & b & 1 \\ a & -b & -1 \\ 0 & 1+b & 1 \end{vmatrix}, \end{aligned}$$

as asserted by (3.11). (Here  $|M|$  denotes the determinant of  $M$ .)  $\diamond$

**Proof** (of Proposition 3.12). Given  $V \in \text{Gr}_{k,n}^{\geq 0}$ , since  $\ker(Z) = W^\perp$  we have

$$f_Z(V^\perp \cap W) = Z((V^\perp \cap W)^\perp) = Z(V + W^\perp) = Z(V) = \tilde{Z}(V).$$

Thus the image  $f_Z(\mathcal{B}_{n,k,1}(W))$  equals  $\mathcal{A}_{n,k,1}(Z)$ , and the result follows from Lemma 3.10.  $\blacksquare$

**3.3. A hypothetical intrinsic description of the amplituhedron.** We now give a description of the amplituhedron  $\mathcal{B}_{n,k,1}(W)$  which does not mention  $\text{Gr}_{k,n}^{\geq 0}$ . This description will extend to  $\mathcal{B}_{n,k,m}(W)$  for  $m > 1$  if part (i) of the following problem has a positive answer.

**Problem 3.14.** Let  $V \in \text{Gr}_{l,n}$ , and  $k \geq l$ .

- (i) If  $\text{var}(v) \leq k-1$  for all  $v \in V$ , can we extend  $V$  to an element of  $\text{Gr}_{k,n}^{\geq 0}$ ?
- (ii) If  $\overline{\text{var}}(v) \leq k-1$  for all  $v \in V \setminus \{0\}$ , can we extend  $V$  to an element of  $\text{Gr}_{k,n}^{\geq 0}$ ?

**Lemma 3.15.** For  $W \in \text{Gr}_{k+m,n}^{\geq 0}$ , we have

$$\mathcal{B}_{n,k,m}(W) \subseteq \{X \in \text{Gr}_m(W) : \overline{\text{var}}(v) \geq k \text{ for all } v \in X \setminus \{0\}\}.$$

If Problem 3.14(i) has a positive answer, then equality holds.

In calculating  $\overline{\text{var}}(v)$  for  $v \in X$  (where  $X \in \text{Gr}_m(W)$ ), we regard  $v$  as a vector in  $\mathbb{R}^n$ . We remark that for any  $X \in \text{Gr}_m(W)$ , since  $W$  is totally positive we have  $\overline{\text{var}}(v) \leq k+m-1$  for all  $v \in X \setminus \{0\}$  by Theorem 3.4(ii).

**Proof.** The first part follows from Theorem 3.4(i). Now suppose that Problem 3.14(i) has a positive answer. Let  $X \in \text{Gr}_m(W)$  with  $\overline{\text{var}}(v) \geq k$  for all  $v \in X \setminus \{0\}$ . Then  $\text{var}(w) \leq n-k-1$  for all  $w \in \text{alt}(X) \setminus \{0\}$  by Lemma 3.3(i). Hence we can extend  $\text{alt}(X)$  to an element of  $\text{Gr}_{n-k,n}^{\geq 0}$ , which by Lemma 3.3(ii) we can write as  $\text{alt}(V^\perp)$  for some  $V \in \text{Gr}_{k,n}^{\geq 0}$ . Since  $\text{alt}(X) \subseteq \text{alt}(V^\perp)$ , we have  $X \subseteq V^\perp$ , whence  $X \subseteq V^\perp \cap W$ . Since  $\dim(X) = m = \dim(V^\perp \cap W)$ , we have  $X = V^\perp \cap W \in \mathcal{B}_{n,k,m}(W)$ .  $\blacksquare$

We remark that showing equality holds in Lemma 3.15 does not resolve Problem 3.14(i), because  $W \in \text{Gr}_{k+m,n}$  is not arbitrary, but is required to be totally positive.

By [Kar16, Lemma 4.1], both parts (i) and (ii) of Problem 3.14 have a positive answer in the case  $m = 1$ . Hence from Lemma 3.15 we obtain the following description of  $\mathcal{B}_{n,k,1}(W)$ , which will be important for us in our study of the structure of  $\mathcal{B}_{n,k,1}(W)$ .

**Corollary 3.16.** *For  $W \in \text{Gr}_{k+1,n}^{>0}$ , we have*

$$\mathcal{B}_{n,k,1}(W) = \{w \in \mathbb{P}(W) : \overline{\text{var}}(w) = k\} \subseteq \mathbb{P}(W).$$

**Remark 3.17.** We give some background on Problem 3.14. Lam [Lam16b, Section 15] considered images  $\tilde{Z}(\text{Gr}_{k,n}^{\geq 0}) \subseteq \text{Gr}_{k,k+m}$  of the totally nonnegative Grassmannian induced by linear maps  $Z : \mathbb{R}^n \rightarrow \mathbb{R}^{k+m}$  more general than those appearing in the construction of amplituhedra (where  $Z$  is represented by a matrix with positive maximal minors). We still need some positivity condition on  $Z$  so that the image  $\tilde{Z}(\text{Gr}_{k,n}^{\geq 0})$  is contained in  $\text{Gr}_{k,k+m}$ , i.e. that

$$(3.18) \quad \dim(Z(V)) = k \quad \text{for all } V \in \text{Gr}_{k,n}^{\geq 0}.$$

Lam [Lam16b, Proposition 15.2] showed that (3.18) holds if the row span of  $Z$  has a subspace in  $\text{Gr}_{k,n}^{>0}$ , in which case he calls  $\tilde{Z}(\text{Gr}_{k,n}^{\geq 0})$  a *(full) Grassmann polytope*, since in the case  $k = 1$  Grassmann polytopes are precisely polytopes in  $\mathbb{P}^m$ . It is an open problem to determine whether Lam's condition is equivalent to (3.18). It turns out that this problem is equivalent to Problem 3.14(ii); see [Kar16, Theorem 4.2] and the discussion which follows it. Thus Problem 3.14(ii) is fundamental to the study of Grassmann polytopes. In the current context, we are more concerned with part (i) of Problem 3.14, though part (ii) will reappear in Section 3.4.

We now translate the conjectural description of Lemma 3.15 into one in terms of Plücker coordinates. Note that this conjectural description is reminiscent of a description of the amplituhedron which was conjectured by Arkani-Hamed, Thomas, and Trnka [AHTT]. We thank Jara Trnka for telling us about this conjecture.

**Proposition 3.19.** *Let  $Z$  be a  $(k+m) \times n$  matrix whose maximal minors are all positive, with row span  $W \in \text{Gr}_{k+m,n}^{>0}$ . Define the open subset of  $\text{Gr}_m(W)$*

$$\mathcal{G} := \{X \in \text{Gr}_m(W) : \text{var}(v) \geq k \text{ for all } v \in X \setminus \{0\}\}.$$

*We have  $\mathcal{B}_{n,k,m}(W) \subseteq \overline{\mathcal{G}}$ , and if Problem 3.14(i) has a positive answer then equality holds. Independently of Problem 3.14(i), we can describe  $\mathcal{G}$  in terms of Plücker coordinates:*

$$\mathcal{G} = \left\{ X \in \text{Gr}_m(W) : \begin{array}{l} \text{var}((( -1)^{|I \cap [j]|} \Delta_{I \cup \{j\}}(X))_{j \in [n] \setminus I}) \geq k \text{ for all } I \in \binom{[n]}{m-1} \text{ such} \\ \text{that the sequence } (\Delta_{I \cup \{j\}}(X))_{j \in [n] \setminus I} \text{ is not identically zero} \end{array} \right\}.$$

*We can translate this description to  $\mathcal{A}_{n,k,m}(Z)$  as follows. Let  $\mathcal{F} := f_Z(\mathcal{G})$  be the image of  $\mathcal{G}$  under the isomorphism  $f_Z$  from Lemma 3.10, and  $z_1, \dots, z_n \in \mathbb{R}^{k+m}$  be the columns of  $Z$ . Then*

$$\mathcal{F} = \left\{ \langle y_1, \dots, y_k \rangle : \begin{array}{l} \text{var}((\det([y_1 \mid \dots \mid y_k \mid z_{i_1} \mid \dots \mid z_{i_{m-1}} \mid z_j]))_{j \in [n] \setminus I}) \geq k \\ \text{for all } I = \{i_1 < \dots < i_{m-1}\} \in \binom{[n]}{m-1} \text{ such that} \\ \text{this sequence of minors is not identically zero} \end{array} \right\}.$$

*We have  $\mathcal{A}_{n,k,m}(Z) \subseteq \overline{\mathcal{F}}$ , and if Problem 3.14(i) has a positive answer then equality holds.*

As stated, the descriptions of  $\mathcal{F}$  and  $\mathcal{G}$  require checking  $\binom{n}{m-1}$  sequences of minors. Does it suffice to check fewer sequences?

**Remark 3.20.** For all  $X \in \text{Gr}_m(W)$ , we have  $\overline{\text{var}}(v) \leq k + m - 1$  for all  $v \in X \setminus \{0\}$  by Theorem 3.4(ii), whence  $\overline{\text{var}}((\Delta_{I \cup \{j\}}(X))_{j \in [n] \setminus I}) \leq k$  for all  $I \in \binom{[n]}{m-1}$  such that the sequence  $(\Delta_{I \cup \{i\}}(X))_{j \in [n] \setminus I}$  is not identically zero (see [Kar16, Theorem 3.1(ii)]). Note that the sequence  $(\Delta_{I \cup \{j\}}(X))_{j \in [n] \setminus I}$  is the one appearing in the description of  $\mathcal{G}$ , without the sign  $(-1)^{|I \cap [j]|}$ . For the sequence in the description of  $\mathcal{F}$ , this sign change corresponds to moving  $z_j$  from the right end of the matrix into its proper relative position in the submatrix of  $Z$ . If  $I \in \binom{[n]}{m-1}$  satisfies the condition that  $(-1)^{|I \cap [j]|}$  is the same for all  $j \in [n] \setminus I$  (called *Gale's evenness condition* [Gal63]), then these two sequences are the same up to multiplication by  $\pm 1$ .

**Example 3.21.** Let  $(n, k, m) := (5, 1, 2)$ , and  $Z : \mathbb{R}^5 \rightarrow \mathbb{R}^3$  be given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} =: [z_1 \mid z_2 \mid z_3 \mid z_4 \mid z_5],$$

whose  $3 \times 3$  minors are all positive. In this case  $\mathcal{A}_{5,1,2}(Z)$  is a convex pentagon in  $\mathbb{P}^2 = \text{Gr}_1(\mathbb{R}^3)$ . Given  $y \in \mathbb{R}^3$ , Proposition 3.19 says that  $\langle y \rangle \in \mathcal{F}$  if and only if each sequence  $(\det([y \mid z_i \mid z_j]))_{j \in [5] \setminus \{i\}}$ , for  $i \in [5]$ , is either identically zero or changes sign at least once. For example, if  $i = 3$ , this sequence is

$$\left( \begin{vmatrix} y_1 & 0 & 1 \\ y_2 & 0 & 0 \\ y_3 & 1 & 0 \end{vmatrix}, \begin{vmatrix} y_1 & 0 & 0 \\ y_2 & 0 & 1 \\ y_3 & 1 & 0 \end{vmatrix}, \begin{vmatrix} y_1 & 0 & 1 \\ y_2 & 0 & -1 \\ y_3 & 1 & 1 \end{vmatrix}, \begin{vmatrix} y_1 & 0 & 3 \\ y_2 & 0 & -2 \\ y_3 & 1 & 1 \end{vmatrix} \right) = (y_2, -y_1, y_1 + y_2, 2y_1 + 3y_2).$$

Geometrically, the sequence corresponding to  $i$  records where the point  $\langle y \rangle \in \mathbb{P}^2$  lies in relation to each of the 4 lines joining vertex  $i$  to another vertex of the pentagon. If this sequence does not change sign, then  $\langle y \rangle$  lies on the same side of all 4 of these lines, i.e. the line segment between  $\langle y \rangle$  and vertex  $i$  does not intersect the interior of the pentagon. We see that  $\mathcal{F}$  is the interior of the pentagon. In general,  $\mathcal{F}$  is the interior of  $\mathcal{A}_{n,k,m}(Z)$  if  $k = 1$ .  $\diamond$

**Remark 3.22.** If  $m = 1$ , then  $\mathcal{G}$  is the interior of  $\mathcal{B}_{n,k,m}(W)$  (and  $\mathcal{F}$  is the interior of  $\mathcal{A}_{n,k,m}(Z)$ ) by Proposition 7.2. We observed in Example 3.21 that this also holds if  $k = 1$ . Does it hold in general?

**Proof** (of Proposition 3.19). The fact that  $\mathcal{B}_{n,k,m}(W) \subseteq \overline{\mathcal{G}}$  follows from these two facts:

- $\text{Gr}_{k,n}^{\geq 0} = \overline{\text{Gr}_{k,n}^{>0}}$  [Pos, Section 17];
- if  $V \in \text{Gr}_{k,n}^{>0}$ , then  $V^\perp \cap W \in \mathcal{G}$  (by Theorem 3.4(ii)).

Conversely, if Problem 3.14(i) has a positive answer, then

$$(3.23) \quad \mathcal{B}_{n,k,m}(W) = \{X \in \text{Gr}_m(W) : \overline{\text{var}}(v) \geq k \text{ for all } v \in X \setminus \{0\}\}$$

by Lemma 3.15. Since  $\overline{\mathcal{G}}$  is contained in the right-hand side of (3.23), we get  $\mathcal{B}_{n,k,m}(W) = \overline{\mathcal{G}}$ .

Now we describe  $\mathcal{G}$  in terms of Plücker coordinates. By Lemma 3.3(i), we have

$$\mathcal{G} = \{X \in \text{Gr}_m(W) : \overline{\text{var}}(v) \leq n - k - 1 \text{ for all } v \in \text{alt}(X) \setminus \{0\}\}.$$



Theorem 3.1(ii) of [Kar16] states that for  $X' \in \text{Gr}_{m,n}$ , we have  $\overline{\text{var}}(v) \leq n - k - 1$  for all  $v \in X' \setminus \{0\}$  if and only if  $\overline{\text{var}}((\Delta_{I \cup \{j\}}(X'))_{j \in [n] \setminus I}) \leq n - k - m$  for all  $I \in \binom{[n]}{m-1}$  such that the sequence  $(\Delta_{I \cup \{j\}}(X'))_{j \in [n] \setminus I}$  is not identically zero. Also by Lemma 3.3(i), we have  $\overline{\text{var}}(p) + \text{var}(\text{alt}(p)) = n - m$  for all nonzero  $p \in \mathbb{R}^{[n] \setminus I}$  ( $I \in \binom{[n]}{m-1}$ ), where  $\text{alt}$  acts on  $\mathbb{R}^{[n] \setminus I}$  by changing the sign of every second component. We get

$$\mathcal{G} = \left\{ X \in \text{Gr}_m(W) : \begin{array}{l} \text{var}(\text{alt}(\Delta_{I \cup \{j\}}(\text{alt}(X)))_{j \in [n] \setminus I}) \geq k \text{ for all } I \in \binom{[n]}{m-1} \text{ such} \\ \text{that the sequence } (\Delta_{I \cup \{j\}}(X))_{j \in [n] \setminus I} \text{ is not identically zero} \end{array} \right\}.$$

In order to obtain the desired description of  $\mathcal{G}$ , it remains to show that given  $I \in \binom{[n]}{m-1}$  such that the sequence  $(\Delta_{I \cup \{j\}}(X))_{j \in [n] \setminus I}$  is not identically zero, we have

$$(3.24) \quad \text{var}(\text{alt}(\Delta_{I \cup \{j\}}(\text{alt}(X)))_{j \in [n] \setminus I}) = \text{var}((( -1)^{|I \cap [j]|} \Delta_{I \cup \{j\}}(X))_{j \in [n] \setminus I}).$$

To this end, write  $I = \{i_1, \dots, i_{m-1}\} \in \binom{[n]}{m-1}$ . Then for  $j \in [n] \setminus I$ , component  $j$  of  $\text{alt}(\Delta_{I \cup \{j\}}(\text{alt}(X)))_{j' \in [n] \setminus I} \in \mathbb{R}^{[n] \setminus I}$  equals

$$(-1)^{|([n] \setminus I) \cap [j]| - 1} (-1)^{(i_1 - 1) + \dots + (i_{m-1} - 1) + (j - 1)} \Delta_{I \cup \{j\}}(X) = \epsilon_I (-1)^{|I \cap [j]|} \Delta_{I \cup \{j\}}(X),$$

where  $\epsilon_I := (-1)^{(i_1 - 1) + \dots + (i_{m-1} - 1)} = \pm 1$  does not depend on  $j$ . This gives (3.24).

Finally, we translate this description of  $\mathcal{G}$  into one for  $\mathcal{F}$ ; the rest follows from Lemma 3.10. By (3.11), we get

$$\mathcal{F} = \left\{ \langle y_1, \dots, y_k \rangle : \begin{array}{l} \text{var}((( -1)^{|I \cap [j]|} \det([y_1 \mid \dots \mid y_k \mid Z_{I \cup \{j\}}]))_{j \in [n] \setminus I}) \geq k \\ \text{for all } I = \{i_1 < \dots < i_{m-1}\} \in \binom{[n]}{m-1} \text{ such that} \\ \text{this sequence of minors is not identically zero} \end{array} \right\},$$

where  $Z_J$  denotes the submatrix of  $Z$  with columns  $J$ , for  $J \subseteq [n]$ . We can move the column of  $Z_{I \cup \{j\}}$  labeled by  $j$  to the right end of the matrix; this introduces a sign  $(-1)^{|I \cap (j, n]|}$  in the determinant. Since  $(-1)^{|I \cap [j]|} (-1)^{|I \cap (j, n]|} = (-1)^{|I|}$ , which does not depend on  $j \in [n] \setminus I$ , the stated description of  $\mathcal{F}$  follows.  $\blacksquare$

**3.4. Removing  $k$  from the definition of the amplituhedron.** In the definition of  $\mathcal{B}_{n,k,1}(W)$  (Definition 3.8), if we let  $W$  and  $k$  vary, then we obtain the following object:

$$(3.25) \quad \widehat{\mathcal{B}}_{m,n} := \bigcup_{0 \leq k \leq n-m} \{V^\perp \cap W : V \in \text{Gr}_{k,n}^{\geq 0}, W \in \text{Gr}_{k+m,n}^{>0}\} \subseteq \text{Gr}_{m,n}.$$

By Theorem 3.4, we have

$$(3.26) \quad \widehat{\mathcal{B}}_{m,n} \subseteq \bigcup_{0 \leq k \leq n-m} \{X \in \text{Gr}_{m,n} : k \leq \overline{\text{var}}(v) \leq k + m - 1 \text{ for all } v \in X \setminus \{0\}\},$$

and equality holds if both parts of Problem 3.14 have a positive answer. (The proof is similar to that of Lemma 3.15.) If  $m = 1$ , then Problem 3.14 has a positive answer, and so  $\widehat{\mathcal{B}}_{1,n} = \text{Gr}_{1,n}$ . However  $\widehat{\mathcal{B}}_{m,n} \neq \text{Gr}_{m,n}$  for  $m \geq 2$ .

Our motivation for considering  $\widehat{\mathcal{B}}_{m,n}$  is that in the BCFW recursion [BCFW05], which conjecturally provides a “triangulation” of  $\mathcal{A}_{n,k,4}(Z)$ , the parameter  $k$  is allowed to vary.

This suggests that the diagrams appearing in the BCFW recursion might label pieces of an object which somehow encompasses  $\mathcal{A}_{n,k,4}(Z)$  for all  $k$ . Could  $\widehat{\mathcal{B}}_{4,n}$  be such an object?

As a first result about  $\widehat{\mathcal{B}}_{m,n}$ , we will show that the union in (3.25) is disjoint if  $m \geq 1$ . It suffices to prove that the union in (3.26) is disjoint, which follows from the lemma below. It also follows that the union in (3.26) consists of the elements of  $\text{Gr}_{m,n}$  whose range of  $\overline{\text{var}}(\cdot)$  is minimum. That is, for  $X \in \text{Gr}_{m,n}$  with  $m \geq 1$ , we have  $\sup_{v,w \in X \setminus \{0\}} \overline{\text{var}}(v) - \overline{\text{var}}(w) \geq m - 1$ , and equality holds if and only if  $X$  is contained in the union in (3.26).

**Lemma 3.27.** *Suppose that  $X \in \text{Gr}_{m,n}$  with  $m \geq 1$ , and  $k \geq 0$  such that  $k \leq \overline{\text{var}}(v) \leq k + m - 1$  for all  $v \in X \setminus \{0\}$ . Then  $\overline{\text{var}}(v) = k + m - 1$  for some  $v \in X \setminus \{0\}$  and  $\overline{\text{var}}(w) = k$  for some  $w \in X \setminus \{0\}$ .*

**Proof.** Let  $v^{(1)}, \dots, v^{(m)}$  be the rows of an  $m \times n$  matrix whose rows span  $X$ , after it has been put into reduced row echelon form. That is, if  $i_1 < \dots < i_m$  index the pivot columns of this matrix, then we have  $v_{is}^{(r)} = \delta_{r,s}$  for all  $r, s \in [m]$ , and  $v_j^{(r)} = 0$  for all  $r \in [m]$  and  $j < i_r$ . Let  $v := v^{(m)}$ , and note that  $\overline{\text{var}}(v) = \overline{\text{var}}(v|_{[i_m,n]}) + i_m - 1$ . Now let

$$w := v + t \sum_{r=1}^{m-1} \epsilon_r v^{(r)},$$

where  $t > 0$  is sufficiently small that in positions  $j \in [i_m, n]$  where  $v$  is nonzero,  $w_j$  has the same sign as  $v_j$ , and  $\epsilon_r \in \{1, -1\}$  is chosen to be 1 precisely if  $|\{i \in \mathbb{Z} : i_r < i < i_{r+1}\}|$  is even. Then

$$k \leq \overline{\text{var}}(w) \leq \overline{\text{var}}(v|_{[i_m,n]}) + i_m - m = \overline{\text{var}}(v) - m + 1 \leq (k + m - 1) - m + 1 = k,$$

and hence equality holds everywhere above. ■

#### 4. A BCFW-LIKE RECURSION FOR $m = 1$

In the case that  $m = 4$ , the *BCFW recursion* (named after Britto, Cachazo, Feng, and Witten [BCF05, BCFW05]) can be viewed as a procedure which outputs a subset of cells of  $\text{Gr}_{k,n}^{\geq 0}$  whose images conjecturally “triangulate” the amplituhedron  $\mathcal{A}_{n,k,4}(Z)$  [AHT14]. The procedure is described in [AHBC<sup>+</sup>16, Section 2] as an operation on plabic graphs.

In this section we give an  $m = 1$  analogue of the BCFW recursion, which naturally leads us to a subset of cells of  $\text{Gr}_{k,n}^{\geq 0}$  that we call the  $m = 1$  *BCFW cells* of  $\text{Gr}_{k,n}^{\geq 0}$ . We remark that unlike the recursion for  $m = 4$  as described in [AHBC<sup>+</sup>16], there is no ‘shift’ to the decorated permutation involved. These cells can be easily described in terms of their  $\mathbb{J}$ -diagrams, positroids, or decorated permutations. As we will show in Section 5, the images of these cells “triangulate” the  $m = 1$  amplituhedron  $\mathcal{A}_{n,k,1}(Z)$ . More specifically, they each map injectively into  $\mathcal{A}_{n,k,1}(Z)$ , and their images are disjoint and together form a dense subset of  $\mathcal{A}_{n,k,1}(Z)$ . In fact, we will show that the BCFW cells in  $\text{Gr}_{k,n}^{\geq 0}$  plus the cells in their boundaries give rise to a cell decomposition of  $\mathcal{A}_{n,k,1}(Z)$ .

**Definition 4.1.** The *BCFW recursion* for  $m = 1$  is defined as follows.

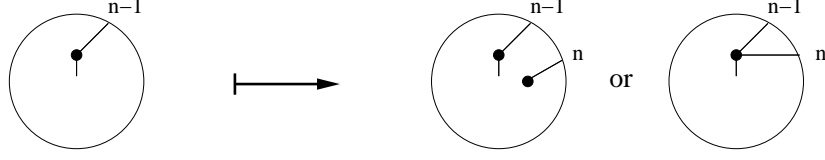


FIGURE 6. The  $m = 1$  BCFW recursion: add a new vertex  $n$ , which is incident either to a black lollipop or to the edge adjacent to vertex  $n - 1$ .

- We start from the plabic graph with one boundary vertex, incident to a black lollipop. This is our graph for  $n = 1$  (and  $k = 0$ ).
- Given a graph with  $n - 1$  boundary vertices which is produced by our recursion, we perform one of the following operations: we add a new boundary vertex  $n$  which is incident either to a black lollipop, or to the edge adjacent to boundary vertex  $n - 1$ . See Figure 6. The first operation preserves the  $k$  statistic, while the second operation increases it by 1.
- We refer to the set of all plabic graphs with fixed  $n$  and  $k$  statistics that we can produce this way as the  $m = 1$  BCFW cells of  $\text{Gr}_{k,n}^{\geq 0}$ .

See Figure 7 for all  $m = 1$  BCFW cells of  $\text{Gr}_{k,n}^{\geq 0}$  with  $n \leq 4$ .

The following lemma is easy to verify by inspection, using the bijections between plabic graphs,  $\mathbb{J}$ -diagrams, and decorated permutations which we gave in Section 2.

**Lemma 4.2.** *The  $m = 1$  BCFW cells of  $\text{Gr}_{k,n}^{\geq 0}$  are indexed by the  $\mathbb{J}$ -diagrams of type  $(k, n)$  such that each of the  $k$  rows contains a unique  $+$ , which is at the far right of the row. The decorated permutation of such a  $\mathbb{J}$ -diagram  $D$  can be written in cycle notation as*

$$\pi(D) = (i_1, i_1 - 1, i_1 - 2, \dots, 1)(i_2, i_2 - 1, \dots, i_1 + 1) \cdots (n, n - 1, \dots, i_{n-k-1} + 1),$$

where  $i_1 < i_2 < \cdots < i_{n-k-1} < i_{n-k} \leq n$  label the horizontal steps of the southeast border of  $D$  (read northeast to southwest), and all fixed points are colored black. In particular, the number of  $m = 1$  BCFW cells of  $\text{Gr}_{k,n}^{\geq 0}$  equals  $\binom{n-1}{k}$ .

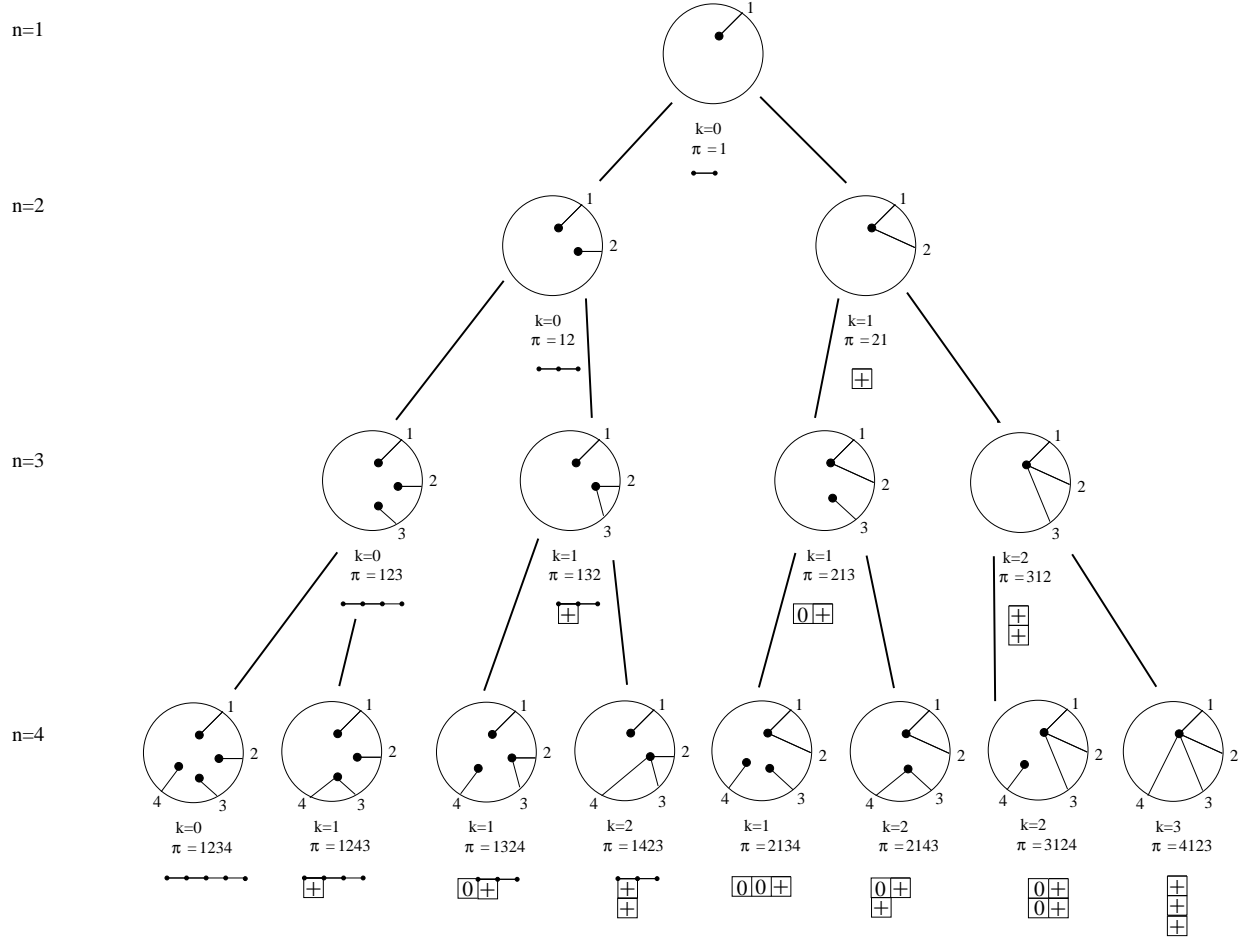
## 5. $\mathcal{A}_{n,k,1}$ AS A SUBCOMPLEX OF THE TOTALLY NONNEGATIVE GRASSMANNIAN

In this section we show that the amplituhedron  $\mathcal{B}_{n,k,1}(W)$  (for  $W \in \text{Gr}_{k+1,n}^{\geq 0}$ ) is isomorphic to a subcomplex of  $\text{Gr}_{k,n}^{\geq 0}$ . We begin by defining a stratification of  $\mathcal{B}_{n,k,1}(W)$ , whose strata are indexed by sign vectors  $\overline{\text{PSign}}_{n,k,1}$  and have a natural poset structure. We will also define a poset of certain  $\mathbb{J}$ -diagrams  $\overline{\mathcal{D}}_{n,k,1}$  and a poset of positroids  $\overline{\mathcal{M}}_{n,k,1}$  and show that all three posets are isomorphic. Finally we will give an isomorphism between the amplituhedron  $\mathcal{B}_{n,k,1}(W)$  and the subcomplex of  $\text{Gr}_{k,n}^{\geq 0}$  indexed by the cells associated to  $\overline{\mathcal{D}}_{n,k,1}$ , which induces an isomorphism on the posets of closures of strata, ordered by containment.

Recall that by Corollary 3.16, we have

$$\mathcal{B}_{n,k,1}(W) = \{w \in W \setminus \{0\} : \overline{\text{var}}(w) = k\} \subseteq \mathbb{P}(W),$$

where we identify a nonzero vector in  $W$  with the line it spans in  $\mathbb{P}(W)$ . We define a stratification of  $\mathcal{B}_{n,k,1}(W)$  using sign vectors.

FIGURE 7. The  $m = 1$  BCFW-style recursion for  $n = 1, 2, 3, 4$ .

**Definition 5.1.** Let  $\overline{\text{Sign}}_{n,k,1}$  denote the set of nonzero sign vectors  $\sigma \in \{0, +, -\}^n$  with  $\overline{\text{var}}(\sigma) = k$ , such that if  $i \in [n]$  indexes the first nonzero component of  $\sigma$ , then  $\sigma_i = (-1)^{i-1}$ . (Equivalently, the first nonzero component of  $\text{alt}(\sigma)$  equals  $+$ .) Also let  $\overline{\mathbb{P}\text{Sign}}_{n,k,1}$  denote the set of nonzero sign vectors  $\sigma \in \{0, +, -\}^n$  with  $\overline{\text{var}}(\sigma) = k$ , modulo multiplication by  $\pm 1$ . We let  $\text{Sign}_{n,k,1}$  and  $\mathbb{P}\text{Sign}_{n,k,1}$  be the subsets of  $\overline{\text{Sign}}_{n,k,1}$  and  $\overline{\mathbb{P}\text{Sign}}_{n,k,1}$  consisting of vectors with no zero components.

**Definition 5.2.** We stratify the amplituhedron  $\mathcal{B}_{n,k,1}(W)$  by  $\overline{\mathbb{P}\text{Sign}}_{n,k,1}$ , i.e. its strata are  $\mathcal{B}_\sigma(W) := \{w \in W \setminus \{0\} : \text{sign}(w) = \pm\sigma\}$  for  $\sigma \in \overline{\mathbb{P}\text{Sign}}_{n,k,1}$ . All strata are nonempty by Lemma 3.7(i). The strata are partially ordered by containment of closures, i.e.  $\mathcal{B}_\sigma(W) \leq \mathcal{B}_\tau(W)$  if and only if  $\overline{\mathcal{B}_\sigma(W)} \subseteq \overline{\mathcal{B}_\tau(W)}$ .

This stratification for  $\mathcal{B}_{5,2,1}(W)$  is shown in Figure 10. We will show in Theorem 5.17 and Theorem 6.16 that the partial order on strata corresponds to a very natural partial order on  $\overline{\mathbb{P}\text{Sign}}_{n,k,1}$ , which we now describe.

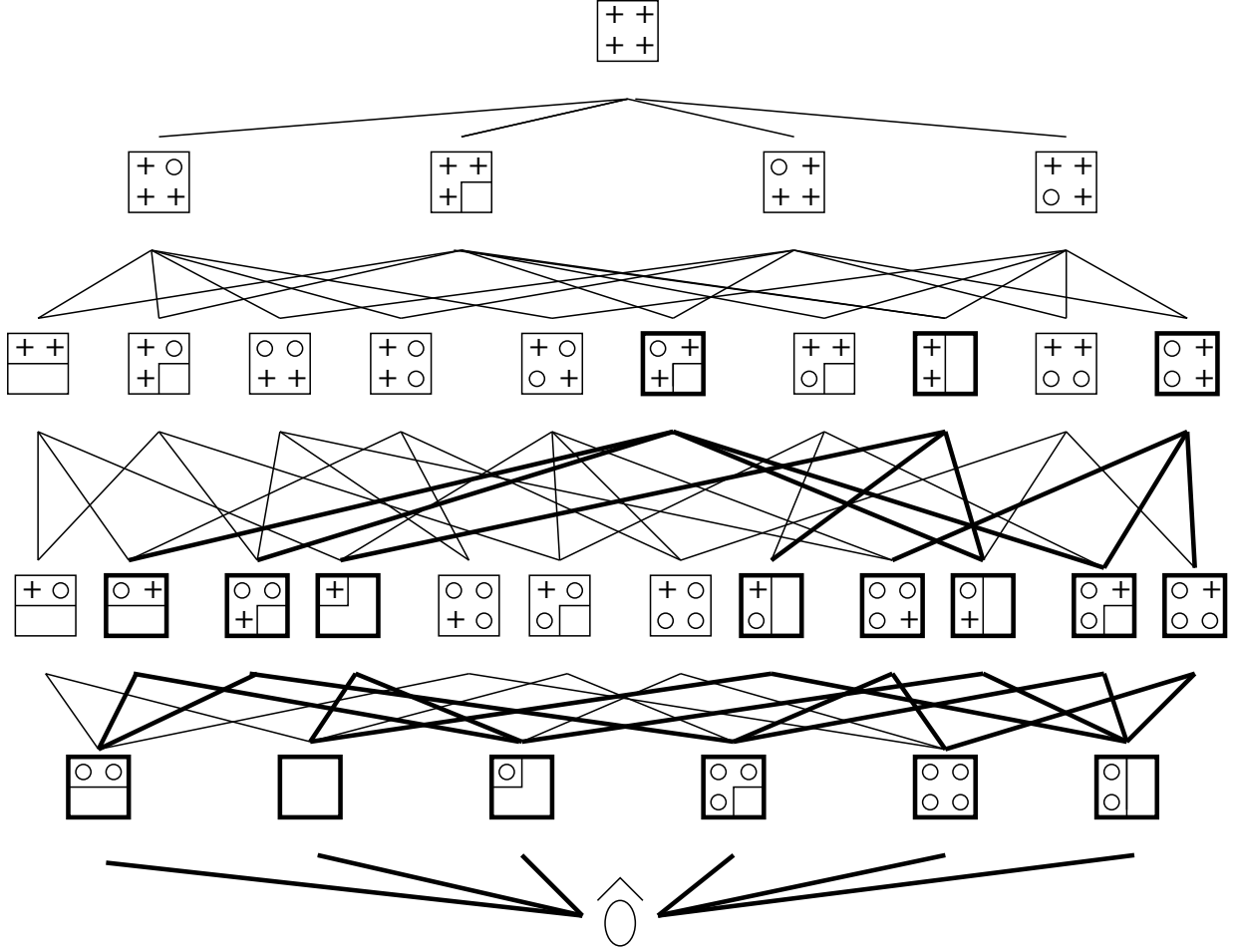


FIGURE 8. The poset  $Q_{2,4}$  of cells of  $\text{Gr}_{2,4}^{\geq 0}$ , where each cell is identified with the corresponding  $\mathbb{J}$ -diagram. The bold edges indicate the subcomplex (an induced subposet) which gets identified with the amplituhedron  $\mathcal{A}_{4,2,1}(Z)$ .

**Definition 5.3.** We define a partial order on the set of sign vectors  $\{0, +, -\}^n$  as follows:  $\sigma \leq \tau$  if and only if  $\sigma_i = \tau_i$  for all  $i \in [n]$  such that  $\sigma_i \neq 0$ . Equivalently,  $\sigma \leq \tau$  if and only if we can obtain  $\sigma$  by setting some components of  $\tau$  to 0. This gives a partial order on  $\overline{\text{Sign}}_{n,k,1}$  by restriction. And for nonzero sign vectors  $\sigma, \tau$  representing elements in  $\overline{\mathbb{P}\text{Sign}}_{n,k,1}$ , we say that  $\sigma \leq \tau$  if and only if  $\sigma \leq \tau$  or  $\sigma \leq -\tau$  in  $\overline{\text{Sign}}_{n,k,1}$ .

For example, we have  $(+, 0, +, 0, +) \leq (+, -, +, +, +)$ , but  $(+, 0, +, 0, +) \not\leq (+, -, 0, +, +)$ . Figure 10 shows  $\overline{\text{Sign}}_{5,2,1}$  as labels of the bounded faces of a hyperplane arrangement.

We will now show that  $\overline{\text{Sign}}_{n,k,1}$  and  $\overline{\mathbb{P}\text{Sign}}_{n,k,1}$  are isomorphic as posets. Our reason for using both posets is as follows. Since  $\mathcal{B}_{n,k,1}(W)$  is a subset of the projective space  $\mathbb{P}(W)$ , the sign vectors used to index strata should be considered modulo multiplication by  $\pm 1$ , which leads us naturally to  $\overline{\mathbb{P}\text{Sign}}_{n,k,1}$ . However, in Theorem 6.16 we will show that  $\mathcal{B}_{n,k,1}(W)$  is isomorphic to the bounded complex of a hyperplane arrangement, and the bounded faces of

this arrangement are labeled by sign vectors *not* considered modulo multiplication by  $\pm 1$ . To prove this result, we will need to work with  $\overline{\text{Sign}}_{n,k,1}$ , which requires a more careful analysis.

**Lemma 5.4.**

- (i) The map  $\overline{\text{Sign}}_{n,k,1} \rightarrow \overline{\mathbb{P}\text{Sign}}_{n,k,1}, \sigma \mapsto \sigma$  is an isomorphism of posets.  
(ii) Conversely, suppose that  $P$  is an induced subposet of  $\{\sigma \in \{0, +, -\}^n \setminus \{0\} : \overline{\text{var}}(\sigma) = k\}$  such that the map  $P \rightarrow \overline{\mathbb{P}\text{Sign}}_{n,k,1}, \sigma \mapsto \sigma$  is an isomorphism of posets. Then  $P$  equals  $\overline{\text{Sign}}_{n,k,1}$  or  $-\overline{\text{Sign}}_{n,k,1}$ .

This says that  $\pm \overline{\text{Sign}}_{n,k,1}$  are the unique liftings of  $\overline{\mathbb{P}\text{Sign}}_{n,k,1}$  to  $\{0, +, -\}^n$  which preserve its poset structure. For example, if  $n = 2, k = 1$ , then the lifting  $P := \{(+, -), (+, 0), (0, +)\}$  does not have the same poset structure as  $\overline{\text{Sign}}_{2,1,1} = \{(+, -), (+, 0), (0, -)\}$ , since  $P$  has two maximal elements, but  $\overline{\text{Sign}}_{2,1,1}$  has the unique maximum  $(+, -)$ .

**Proof.** (i) The map  $\overline{\text{Sign}}_{n,k,1} \rightarrow \overline{\mathbb{P}\text{Sign}}_{n,k,1}, \sigma \mapsto \sigma$  is a bijection and a poset homomorphism. To show that it is a poset isomorphism, we must show that there do not exist  $\sigma, \tau \in \overline{\text{Sign}}_{n,k,1}$  with  $\sigma \leq -\tau$ . Suppose otherwise that such  $\sigma$  and  $\tau$  exist. Let  $\sigma' := \text{alt}(\sigma)$  and  $\tau' := \text{alt}(\tau)$ . By Lemma 3.3(i), we have  $\text{var}(\sigma') = \text{var}(\tau') = n - k - 1$ . Let  $i, j \in [n]$  index the first nonzero components of  $\sigma', \tau'$ , respectively, so that  $\sigma'_i = \tau'_j = +$ . But also by our assumption,  $\sigma'_i = -\tau'_i$ . Since  $j < i$ ,  $-\tau'$  changes sign at least once from  $j$  to  $i$ . This implies  $\text{var}(-\tau') > \text{var}(\sigma')$ , a contradiction.

(ii) Let  $G$  be the graph with vertex set  $\overline{\mathbb{P}\text{Sign}}_{n,k,1}$ , where distinct  $\sigma, \tau \in \overline{\mathbb{P}\text{Sign}}_{n,k,1}$  are adjacent if and only if  $\sigma$  differs in a single component from either  $\tau$  or  $-\tau$ .

*Claim.*  $G$  is connected.

*Proof of Claim.* Let us show that every vertex  $\sigma$  of  $G$  is connected to

$$\hat{\sigma} := (+, -, +, -, \dots, (-1)^{k-1}, (-1)^k, (-1)^k, \dots),$$

i.e.  $\hat{\sigma}$  is the sign vector which alternates in sign on  $[k+1]$ , and is constant thereafter. Take  $i \in [n]$  maximum such that  $\sigma$  alternates in sign on  $[i]$ . If  $i = k+1$ , then  $\sigma = \hat{\sigma}$ . Otherwise, take  $j > i$  minimum with  $\sigma_j \neq \sigma_i$ . That is, on the interval  $[i, j]$ ,  $\sigma$  equals  $(+, +, \dots, +, -)$  up to sign. By performing sign flips at components  $j-1, j-2, \dots, i+1$ , we obtain a sign vector which equals  $(+, -, -, \dots, -)$  on  $[i, j]$ , and hence alternates in sign on  $[i+1]$ . Repeating this procedure for  $i+1, i+2, \dots, k$ , we obtain  $\hat{\sigma}$ . For example, if  $\sigma = (+, +, +, -, -, +, -)$  (where  $n = 7, k = 3$ ), then we obtain  $\hat{\sigma}$  as follows:

$$\sigma \xrightarrow{i=1} (+, -, -, -, -, +, -) \xrightarrow{i=2} (+, -, +, +, +, +, -) \xrightarrow{i=3} (+, -, +, -, -, -, -) = \hat{\sigma}.$$

This proves the claim.  $\square$

Let  $H$  be the corresponding graph for  $P$ , i.e.  $H$  has vertex set  $Q := P \cap \{+, -\}^n$ , and two distinct sign vectors are adjacent in  $H$  if and only if they differ in a single component. Note that  $G$  (respectively  $H$ ) depends only on the poset  $\overline{\mathbb{P}\text{Sign}}_{n,k,1}$  (respectively  $P$ ): two distinct sign vectors are adjacent if and only if they cover a common sign vector in the poset. Since  $P \cong \overline{\mathbb{P}\text{Sign}}_{n,k,1}$  and  $G$  is connected by sign flips, we get that  $H$  is connected by sign flips. Note that we can never flip the first component, which would change sign variation. Hence

all elements of  $Q$  have the same first component. After replacing  $P$  with  $-P$  if necessary, we may assume that  $\sigma_1 = +$  for all  $\sigma \in Q$ .

Let  $\sigma \in P$ . We will show that  $\sigma_i = (-1)^{i-1}$ , where  $i$  indexes the first nonzero component of  $\sigma$ . Since  $\overline{\text{var}}(\sigma) = k$ , we can extend  $\sigma$  to  $\tau \in \{+, -\}^n$  (i.e.  $\tau \geq \sigma$ ) with  $\text{var}(\tau) = k$ . Then  $\tau$  alternates in sign on  $[i]$ , i.e. it equals  $((-1)^{i-1}\sigma_i, (-1)^{i-2}\sigma_i, \dots, \sigma_i)$  on  $[i]$ . Now since  $P \rightarrow \overline{\mathbb{P}\text{Sign}_{n,k,1}}$ ,  $\sigma \mapsto \sigma$  is a poset isomorphism, exactly one of  $\tau, -\tau$  is in  $P$ . If  $-\tau \in P$ , then  $\sigma \not\leq -\tau$  in  $P$  but  $\sigma \leq -\tau$  in  $\overline{\mathbb{P}\text{Sign}_{n,k,1}}$ , a contradiction. Hence  $\tau$  is in  $P$  (and hence in  $Q$ ), whence  $\tau_1 = +$ , i.e.  $\sigma_i = (-1)^{i-1}$ .  $\blacksquare$

We now define a subcomplex of  $\text{Gr}_{k,n}^{\geq 0}$  which will turn out to be isomorphic to  $\mathcal{B}_{n,k,1}(W)$ .

**Definition 5.5.** Let  $\mathcal{D}_{n,k,1}$  (respectively,  $\overline{\mathcal{D}_{n,k,1}}$ ) be the set of  $\mathbb{J}$ -diagrams contained in a  $k \times (n - k)$  rectangle whose rows each have precisely one  $+$  (respectively, at most one  $+$ ), and each  $+$  appears at the right end of its row. For  $D \in \overline{\mathcal{D}_{n,k,1}}$ , we let  $\dim(D) := \dim(S_D)$  be the number of  $+$ 's in  $D$ .

Note that  $\mathcal{D}_{n,k,1}$  indexes the  $m = 1$  BCFW cells of  $\text{Gr}_{k,n}^{\geq 0}$  by Lemma 4.2.

Since  $\mathbb{J}$ -diagrams index the cells of  $\text{Gr}_{k,n}^{\geq 0}$ ,  $\overline{\mathcal{D}_{n,k,1}}$  has a poset structure as a subposet of  $Q_{k,n}$ . However, it is more convenient for us to define our own partial order on  $\overline{\mathcal{D}_{n,k,1}}$ , as follows; we then show in Lemma 5.13 that our partial order agrees with the one coming from  $Q_{k,n}$ , and that our poset on  $\overline{\mathcal{D}_{n,k,1}}$  is in fact an order ideal (a downset) of  $Q_{k,n}$ .

**Definition 5.6.** We define a partial order on  $\overline{\mathcal{D}_{n,k,1}}$ , with the following cover relations  $\prec$ :

- (Type 1) Let  $D \in \overline{\mathcal{D}_{n,k,1}}$ , where  $\dim(D) \geq 1$ , and choose some  $+$  in  $D$  which has no  $+$ 's below it in the same column. We obtain  $D'$  from  $D$  by deleting the box containing that  $+$  and every box below it and in the same column. Then  $D' \prec D$ .
- (Type 2) Let  $D \in \overline{\mathcal{D}_{n,k,1}}$ , where  $\dim(D) \geq 1$ , and choose some  $+$  in  $D$ . We obtain  $D'$  from  $D$  by replacing the  $+$  with a  $0$ . Then  $D' \prec D$ .

We now give a poset isomorphism  $\Omega_{DS} : \overline{\mathcal{D}_{n,k,1}} \rightarrow \overline{\text{Sign}_{n,k,1}}$ .

**Definition 5.7.** Let  $D \in \mathcal{D}_{n,k,1}$  with Young diagram  $Y_\lambda$ , where the steps of the southeast border of  $Y_\lambda$  are labeled from 1 to  $n$ . Then we define  $\sigma(D) \in \overline{\text{Sign}_{n,k,1}}$  recursively by setting:

- $\sigma_1 := +$ ;
- $\sigma_{i+1} = \sigma_i$  if and only if  $i$  is the label of a horizontal step of  $Y_\lambda$ ,  $i = 1, \dots, n - 1$ .

Now let  $D \in \overline{\mathcal{D}_{n,k,1}}$  with Young diagram  $Y_\lambda$ . We obtain  $\widehat{D}$  from  $D$  by putting a  $+$  at the far right of every row which has no  $+$ . To define  $\sigma(D)$ , we first compute  $\sigma(\widehat{D})$  as above, but then for every vertical step  $i$  corresponding to an all-zero row of  $D$ , we set  $\sigma(D)_i = 0$ . This gives a map  $\Omega_{DS} : \overline{\mathcal{D}_{n,k,1}} \rightarrow \overline{\text{Sign}_{n,k,1}}$  defined by  $\Omega_{DS}(D) = \sigma(D)$ .

For examples of this bijection, compare Figure 11 with Figure 10.

**Lemma 5.8.** *The map  $\Omega_{DS}$  from Definition 5.7 is an isomorphism of posets.*

**Proof.** First we show that  $\Omega_{DS}$  is well defined, i.e. given  $D \in \overline{\mathcal{D}_{n,k,1}}$  we have  $\sigma(D)_i = (-1)^{i-1}$ , where  $i \in [n]$  indexes the first nonzero component of  $\sigma(D)$ . Indeed, we have that  $1, \dots, i - 1$

label vertical steps of the southeast border of  $D$  whose rows contain only 0's, and  $i$  labels either a vertical step whose row contains a +, or a horizontal step. Hence when we lift  $D$  to  $\widehat{D}$  in Definition 5.7,  $\sigma(\widehat{D})$  alternates in sign on  $[1, i]$ , whence  $\sigma(D)_i = \sigma(\widehat{D})_i = (-1)^{i-1}$ .

To show that  $\Omega_{DS}$  is a bijection, we describe the inverse map. Let  $\sigma \in \overline{\text{Sign}}_{n,k,1}$ . We want to construct  $D \in \overline{\mathcal{D}}_{n,k,1}$  such that  $\sigma(D) = \sigma$ . We first lift  $\sigma$  to a sign vector  $\widehat{\sigma} \in \text{Sign}_{n,k,1}$  by reading  $\sigma$  from right to left:

- Set  $\widehat{\sigma}_n := (-1)^k$ .
- For  $i = n-1, \dots, 1$ , if  $\sigma_i = 0$ , we set  $\widehat{\sigma}_i := -\widehat{\sigma}_{i+1}$ , and otherwise we set  $\widehat{\sigma}_i := \sigma_i$ .

The reason we set  $\widehat{\sigma}_n := (-1)^k$  is that if  $\tau \in \{+, -\}^n$  with  $\tau \geq \sigma$  and  $\text{var}(\tau) = k$ , then  $\tau_1 = +$  since  $\tau \in \text{Sign}_{n,k,1}$ , which implies that  $\tau_n = (-1)^k$ . The same reasoning implies that  $\widehat{\sigma} \in \text{Sign}_{n,k,1}$ . Now we construct  $D$  as follows. The vertical steps of the southeast border of  $D$  are in bijection with positions  $i$  of  $\widehat{\sigma}$  such that  $\widehat{\sigma}_i = -\widehat{\sigma}_{i+1}$ . In each row with vertical step labeled  $i$ , where  $\sigma_i \neq 0$ , we place a + in the far right. We then fill the remaining boxes with 0's. Note that  $\sigma(D) = \sigma$ . Moreover, we know that 0's of  $\sigma$  must correspond to vertical steps in the southeast border of  $D$ , which uniquely determines  $D$ . Therefore  $\Omega_{DS}$  is a bijection.

We can check that  $\Omega_{DS}$  is a poset homomorphism. It remains to check that its inverse is a poset homomorphism. To this end, let  $\sigma' \triangleleft \sigma$  be a cover relation in  $\overline{\text{Sign}}_{n,k,1}$ , so that  $\sigma'$  is obtained from  $\sigma$  by setting some  $\sigma_i$  to 0. Construct  $D \in \overline{\mathcal{D}}_{n,k,1}$  as above with  $\sigma(D) = \sigma$ . Since  $\sigma_i \neq 0$ , either  $i$  labels a horizontal step in the southeast border of  $D$  and  $i-1$  labels a vertical step (otherwise we would have  $\overline{\text{var}}(\sigma') > \overline{\text{var}}(\sigma)$ ), or  $i$  labels a vertical step and there is a + in that row. In the first case, let  $D' \triangleleft D$  be the Type 1 cover relation obtained by deleting the box containing the lowest + in column of  $D$  labeled by  $i$  (and all boxes below it). In the second case, let  $D' \triangleleft D$  be the Type 2 cover relation obtained by replacing the unique + in the row labeled by  $i$  with a 0. In either case we have  $\sigma(D') = \sigma'$ . ■

We now introduce the positroids corresponding to  $\overline{\mathcal{D}}_{n,k,1}$ .

**Definition 5.9.** Let  $\mathcal{M}_{n,k,1}$  be the set of matroids of rank  $k$  with ground set  $[n]$  which are direct sums  $M_1 \oplus \dots \oplus M_{n-k}$ , where  $E_1 \sqcup \dots \sqcup E_{n-k}$  is a partition of  $[n]$  into nonempty intervals, and  $M_j$  is the uniform matroid of rank  $|E_j| - 1$  with ground set  $E_j$ , for  $j \in [n-k]$ . Let  $\overline{\mathcal{M}}_{n,k,1}$  be the order ideal of matroids of  $\mathcal{M}_{n,k,1}$ , i.e. the set of matroids  $M'$  of rank  $k$  with ground set  $[n]$  such that  $M' \leq M$  for some  $M \in \mathcal{M}_{n,k,1}$ . (The partial order was defined in Definition 2.19.)

We also define  $\mathcal{M}_{n,k,1}^* := \{M^* : M \in \mathcal{M}_{n,k,1}\}$  and  $\overline{\mathcal{M}}_{n,k,1}^* := \{M^* : M \in \overline{\mathcal{M}}_{n,k,1}\}$ , so that  $\overline{\mathcal{M}}_{n,k,1}^*$  is the set of positroids  $M'$  of rank  $n-k$  with ground set  $[n]$  such that  $M' \leq M$  for some  $M \in \mathcal{M}_{n,k,1}^*$ . Since taking duals commutes with direct sum [Oxl11, Proposition 4.2.21],  $\mathcal{M}_{n,k,1}^*$  is the set of matroids of rank  $k$  with ground set  $[n]$  which are direct sums  $M_1 \oplus \dots \oplus M_{n-k}$ , where  $E_1 \sqcup \dots \sqcup E_{n-k}$  is a partition of  $[n]$  into nonempty intervals, and  $M_j$  is the uniform matroid of rank 1 with ground set  $E_j$ , for  $j \in [n-k]$ .

The matroids in  $\overline{\mathcal{M}}_{n,k,1}$  and  $\overline{\mathcal{M}}_{n,k,1}^*$  are in fact positroids; see Definition 5.12. (Alternatively, using Lemma 5.10, we can easily construct a matrix representing any matroid in  $\overline{\mathcal{M}}_{n,k,1}^*$ .) This implies that  $\overline{\mathcal{M}}_{n,k,1}$  indexes an order ideal in  $Q_{k,n}$ , the poset of cells of  $\text{Gr}_{k,n}^{\geq 0}$ .



**Lemma 5.10.** *The matroids in  $\overline{\mathcal{M}_{n,k,1}}$  and  $\overline{\mathcal{M}_{n,k,1}^*}$  correspond to pairs  $(E_1 \sqcup \cdots \sqcup E_{n-k}, C)$ , where*

- $E_1 \sqcup \cdots \sqcup E_{n-k}$  is a partition of  $[n]$  into nonempty intervals and  $C \subseteq [n]$ ;
- for all  $j \in [n - k]$ ,  $E_j \setminus C$  is nonempty; and
- $\max(E_j) \notin C$  whenever  $\max(E_j) \neq n$ .

*The matroid  $M \in \overline{\mathcal{M}_{n,k,1}}$  associated to  $(E_1 \sqcup \cdots \sqcup E_{n-k}, C)$  is the direct sum  $M_1 \oplus \cdots \oplus M_{n-k}$ , where  $M_j$  ( $j \in [n - k]$ ) is the matroid with ground set  $E_j$  such that  $E_j \cap C$  are coloops and the restriction of  $M_j$  to  $E_j \setminus C$  is a uniform matroid of rank  $|E_j \setminus C| - 1$ .*

*And the matroid  $M^* \in \overline{\mathcal{M}_{n,k,1}^*}$  associated to  $(E_1 \sqcup \cdots \sqcup E_{n-k}, C)$  is the direct sum  $M_1^* \oplus \cdots \oplus M_{n-k}^*$ , where  $M_j^*$  ( $j \in [n - k]$ ) is the matroid with ground set  $E_j$  such that  $E_j \cap C$  are loops and the restriction of  $M_j$  to  $E_j \setminus C$  is a uniform matroid of rank 1.*

**Proof.** The description of  $\overline{\mathcal{M}_{n,k,1}}$  follows from the description of  $\overline{\mathcal{M}_{n,k,1}^*}$ , so we prove the latter. In considering  $\overline{\mathcal{M}_{n,k,1}^*}$ , we will use the following claim.

*Claim.* Let  $M$  be a matroid with ground set  $F$  and connected components  $F_1, \dots, F_l$ , and let  $M' \leq M$ . Then each  $F_j$  ( $j \in [l]$ ) is a union of connected components of  $M'$ , and  $M'|_{F_j} \leq M|_{F_j}$ .

*Proof of Claim.* By [Oxl11, Proposition 4.1.2] (see also [ARW16, Proposition 7.2]), elements  $i, j$  of the ground set are in the same connected component of a matroid  $N$  if and only if there exist bases  $B, B'$  of  $N$  with  $B' = (B \setminus \{i\}) \cup \{j\}$ . It follows that if  $i, j \in F$  are in the same connected component of  $M'$ , then they are in the same connected component of  $M$ . The fact that  $M'|_{F_j} \leq M|_{F_j}$  for  $j \in [l]$  follows from Definition 2.18, as long as  $M'|_{F_j}$  and  $M|_{F_j}$  have the same rank. But this is true since  $\sum_{j=1}^l \text{rank}(M'|_{F_j}) = \text{rank}(M') = \text{rank}(M) = \sum_{j=1}^l \text{rank}(M|_{F_j})$ .  $\square$

Now note that if  $M$  is a uniform matroid of rank 1, then the matroids  $M'$  satisfying  $M' \leq M$  are precisely all matroids of rank 1 with the same ground set as  $M$ . Hence by the claim, the elements of  $\overline{\mathcal{M}_{n,k,1}^*}$  are obtained precisely by taking a matroid  $M_1 \oplus \cdots \oplus M_{n-k} \in \mathcal{M}_{n,k,1}$  with ground set  $E_1 \sqcup \cdots \sqcup E_{n-k}$ , and choosing some subset  $C \subseteq [n]$  of the ground set, satisfying  $E_j \setminus C \neq \emptyset$  for all  $j \in [n - k]$ , to turn into loops (i.e. we delete all bases with a nonempty intersection with  $C$ ). The condition that  $\max(E_j) \notin C$  if  $\max(E_j) \neq n$  comes from our convention that a loop which appears ‘between’ two intervals is associated to the interval on its right.  $\blacksquare$

**Remark 5.11.** Using Lemma 5.10, one can write down the generating function for the stratification of  $\mathcal{B}_{n,k,1}(W)$  with respect to dimension; see Corollary 6.21. We will give a different proof of Corollary 6.21, using the fact that  $\mathcal{B}_{n,k,1}(W)$  is isomorphic to the bounded complex of a generic hyperplane arrangement, whose rank generating function is known.

We now give a poset isomorphism  $\Omega_{DM} : \overline{\mathcal{D}_{n,k,1}} \rightarrow \overline{\mathcal{M}_{n,k,1}}$ .

**Definition 5.12.** Given  $D \in \overline{\mathcal{D}_{n,k,1}}$ , we label the southeast border of  $D$  by the numbers 1 through  $n$ . Let the labels of the horizontal steps be denoted by  $h_1, \dots, h_{n-k}$ . Then set  $E_1 := \{1, 2, \dots, h_1\}$ ,  $E_j := \{h_{j-1} + 1, h_{j-1} + 2, \dots, h_j\}$  for  $1 < j < n - k$ , and  $E_{n-k} := \{h_{n-k-1} + 1, h_{n-k-1} + 2, \dots, n\}$ . Let  $C$  be the set of labels of all vertical steps indexing rows of  $D$  with no  $+$ 's. Then  $(E_1 \sqcup \dots \sqcup E_{n-k}, C)$  determines a positroid in  $\overline{\mathcal{M}_{n,k,1}}$  as in Lemma 5.10, which we denote by  $M(D)$ . By inspection, we see that the map  $\Omega_{DM} : D \mapsto M(D)$  is a bijection, and we will denote the inverse by  $\Omega_{DM}^{-1} : M \mapsto D(M)$ .

We observe that  $M(D)$  is precisely the positroid of  $D$  defined in Definition 2.13. In particular, the elements of  $\overline{\mathcal{M}_{n,k,1}}$  and  $\overline{\mathcal{M}_{n,k,1}^*}$  are all positroids. See Figure 9. The white lollipops correspond to coloops and the black lollipops correspond to loops. By considering perfect orientations, it is easy to see that every component of the graph which is not a white lollipop gives rise to a uniform matroid of corank 1.

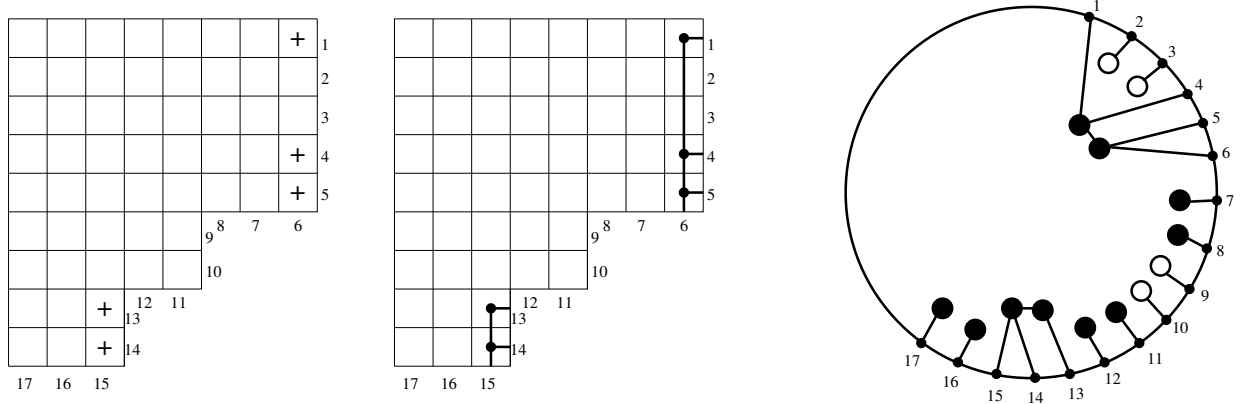


FIGURE 9. Going from a J-diagram in  $\overline{\mathcal{D}_{n,k,1}}$  to the corresponding plabic graph, which in turn determines the positroid in  $\overline{\mathcal{M}_{n,k,1}}$ .

**Lemma 5.13.** *The map  $\Omega_{DM}$  from Definition 5.12 is an isomorphism of posets. In particular,  $\overline{\mathcal{D}_{n,k,1}}$  can be identified with an order ideal of  $Q_{k,n}$ , the poset of cells of  $\text{Gr}_{k,n}^{\geq 0}$ .*

**Proof.** By inspection,  $\Omega_{DM}$  is a bijection and a poset homomorphism. To see that its inverse is a poset homomorphism, let  $M' \triangleleft M$  be a cover relation in  $\overline{\mathcal{M}_{n,k,1}}$ , and take  $D \in \overline{\mathcal{D}_{n,k,1}}$  with  $M = M(D)$ . Let  $E_1 \sqcup \dots \sqcup E_{n-k}$  correspond to  $M$  as in Lemma 5.10. Then we obtain  $M'$  from  $M$  by taking some interval  $E_j$  which contains at least 2 elements which are not coloops, and turn some  $i \in E_j$  into a coloop. If  $i$  is the greatest element of  $E_j$  not already a coloop, then  $i$  labels a horizontal step in the southeast border of  $D$ . In this case, let  $D' \triangleleft D$  be the Type 1 cover relation given by deleting the box containing the lowest  $+$  in column of  $D$  labeled by  $i$  (and all boxes below it). Otherwise  $i$  labels a vertical step in the southeast border of  $D$  whose row contains a  $+$ ; let  $D' \triangleleft D$  be the Type 2 cover relation given by replacing this  $+$  with a 0. Then  $M(D') = M'$ . ■

By composing the bijections from Definitions 5.7 and 5.12, we obtain the following result.

**Corollary 5.14.** *The map  $\Omega_{DS} \circ \Omega_{DM}^{-1} : \overline{\mathcal{M}_{n,k,1}} \rightarrow \overline{\text{Sign}_{n,k,1}}$ ,  $M \mapsto \sigma(M)$  is an isomorphism of posets, where we define  $\sigma(M) := \sigma(D)$  for  $D \in \overline{\mathcal{D}_{n,k,1}}$  with  $M = M(D)$ . We can compute  $\sigma(M)$  from  $M$  as follows. If  $M \in \mathcal{M}_{n,k,1}$  is the direct sum  $M_1 \oplus \cdots \oplus M_{n-k}$  with ground set  $E_1 \sqcup \cdots \sqcup E_{n-k}$ , then  $\sigma(M)$  is uniquely determined by the following two properties:*

- $\sigma(M)_1 = +$ ;
- $\sigma(M)_i = \sigma(M)_{i+1}$  if and only if  $i$  and  $i+1$  are in different blocks of  $E_1 \sqcup \cdots \sqcup E_{n-k}$ .

*And if  $M \in \overline{\mathcal{M}_{n,k,1}}$  is the direct sum  $M_1 \oplus \cdots \oplus M_{n-k}$  with ground set  $E_1 \sqcup \cdots \sqcup E_{n-k}$  and coloops at  $C$ , then  $\sigma(M)$  is given exactly as above, except that  $\sigma(M)_i = 0$  for each  $i \in C$ .*

For example, if  $M \in \mathcal{M}_{8,5,1}$  is associated to  $E_1 \sqcup E_2 \sqcup E_3$  with  $E_1 = \{1, 2, 3\}$ ,  $E_2 = \{4, 5, 6\}$ ,  $E_3 = \{7, 8\}$ , then  $\sigma(M)$  equals  $(+, -, +, +, -, +, +, -)$ .

**Definition 5.15.** Let  $\mathcal{S} := \sqcup_{M \in \overline{\mathcal{M}_{n,k,1}}} S_M = \sqcup_{M \in \mathcal{M}_{n,k,1}} S_M$  be the subcomplex of  $\text{Gr}_{k,n}^{\geq 0}$  corresponding to  $\mathcal{M}_{n,k,1}$ . We define a map (cf. Definition 3.8)

$$\phi_W : \mathcal{S} \rightarrow \mathcal{B}_{n,k,1}(W), \quad V \mapsto V^\perp \cap W.$$

**Proposition 5.16.** *If  $V \in S_M$  for  $M \in \overline{\mathcal{M}_{n,k,1}}$ , then  $V^\perp \cap W \in \mathcal{B}_{\sigma(M)}(W)$ . In other words, the map  $\phi_W$  from Definition 5.15 induces the map  $M \mapsto \sigma(M)$  on strata.*

Technically  $\sigma(M)$  is an element of  $\overline{\text{Sign}_{n,k,1}}$ , while  $\mathcal{B}_{n,k,1}(W)$  is stratified by  $\overline{\mathbb{P}\text{Sign}_{n,k,1}}$ . By Lemma 5.4(i) the map  $\overline{\text{Sign}_{n,k,1}} \rightarrow \overline{\mathbb{P}\text{Sign}_{n,k,1}}$ ,  $\sigma \mapsto \sigma$  is a poset isomorphism, so we need not concern ourselves with this distinction.

**Proof.** We first consider the case that  $M \in \mathcal{M}_{n,k,1}$ . Let us describe the sign vectors of  $V^\perp$  for  $V \in S_M$ . Write  $M = M_1 \oplus \cdots \oplus M_{n-k}$ , where  $E_1 \sqcup \cdots \sqcup E_{n-k}$  is a partition of  $[n]$  into nonempty intervals, and  $M_j$  is the uniform matroid of rank  $|E_j| - 1$  with ground set  $E_j$ , for  $j \in [n-k]$ . By Lemma 3.7(ii), if  $V_j \in S_{M_j}$  then  $\text{sign}(V_j^\perp) = \{\sigma \in \{0, +, -\}^{E_j} : \text{var}(\sigma) \geq |E_j| - 1\} \cup \{0\}$ , i.e.

$$\text{sign}(V_j^\perp) = \{0, (+, -, +, -, \dots), (-, +, -, +, \dots)\}.$$

Hence for  $V \in S_M$ , we have  $\sigma \in \text{sign}(V^\perp)$  if and only if  $\sigma|_{E_j}$  equals 0 or strictly alternates in sign, for all  $j \in [n-k]$ .

Recall from Corollary 3.16 that

$$\mathcal{B}_{n,k,1}(W) = \{w \in W \setminus \{0\} : \overline{\text{var}}(w) = k\} \subseteq \mathbb{P}(W).$$

Note that there is a unique nonzero  $\sigma \in \text{sign}(V^\perp)$  (modulo multiplication by  $\pm 1$ ) with  $\overline{\text{var}}(\sigma) = k$ :  $\sigma$  has no zero components, and  $\sigma_i = \sigma_{i+1}$  if and only if  $i$  and  $i+1$  are in different blocks of  $E_1 \sqcup \cdots \sqcup E_{n-k}$ , for all  $i \in [n-1]$ . This is precisely the definition of  $\sigma(M)$ . Therefore we must have  $V^\perp \cap W \in \mathcal{B}_{\sigma(M)}(W)$ .

Now we consider the general case of sign vectors of  $V'^\perp$ , where  $M' \in \overline{\mathcal{M}_{n,k,1}}$  and  $V' \in S_{M'}$ . We have  $M' \leq M$  for some  $M \in \mathcal{M}_{n,k,1}$ , and  $M'$  is obtained from  $M$  by making some subset  $C \subseteq [n]$  of the ground set coloops. Thus the sign vectors of  $V'^\perp$  are precisely obtained from those of  $V^\perp$  by setting the components indexed by  $C$  to 0. In particular, we again have a unique nonzero  $\sigma' \in \text{sign}(V'^\perp)$  (modulo multiplication by  $\pm 1$ ) with  $\overline{\text{var}}(\sigma') = k$ , which we obtain from  $\sigma(M)$  by setting the components  $C$  to 0. Then  $\sigma' = \sigma(M')$ , and therefore  $V'^\perp \cap W \in \mathcal{B}_{\sigma(M')}(W)$ . ■

**Theorem 5.17.** *The map  $\phi_W$  from Definition 5.15 is a homeomorphism which induces a poset isomorphism on the stratifications of  $\mathcal{S}$  and  $\mathcal{B}_{n,k,1}(W)$ .*

**Proof.** We know from Proposition 5.16 that  $\phi_W$  induces a poset isomorphism on the strata. To show that  $\phi_W$  is a bijection, we construct the inverse map  $\phi_W^{-1} : \mathcal{B}_{n,k,1} \rightarrow \mathcal{S}$  as follows. Given an element of  $\mathcal{B}_{n,k,1}(W)$  spanned by  $w \in W \setminus \{0\}$  (so  $\overline{\text{var}}(w) = k$ ), let  $\sigma$  be either  $\text{sign}(w)$  or  $-\text{sign}(w)$ , whichever is in  $\overline{\text{Sign}}_{n,k,1}$ . Also let  $M \in \overline{\mathcal{M}}_{n,k,1}$  be the positroid such that  $\sigma(M) = \sigma$ , corresponding to  $(E_1 \sqcup \cdots \sqcup E_{n-k}, C)$  in Lemma 5.10. If  $V \in \mathcal{S}$  with  $\phi_W(V) = w$ , then  $V \in S_M$  because  $\phi_W$  induces a poset isomorphism on the strata. Since  $E_j \setminus C \neq \emptyset$  for all  $j \in [n-k]$ ,  $\sigma$  is nonzero when restricted to any interval  $E_j$ . Hence the unique  $V \in \mathcal{S}$  with  $\phi_W(V) = w$  is determined by the conditions

$$V^\perp|_{E_j} = \text{span}(w|_{E_j}) \quad \text{for all } j \in [n-k].$$

Explicitly,  $V^\perp$  has the basis  $w^{(1)}, \dots, w^{(n-k)}$ , where  $w^{(i)}|_{E_j} = \delta_{i,j}w|_{E_j}$  for  $i, j \in [n-k]$ . Thus  $\phi_W$  is invertible, with an inverse which is piecewise polynomial (each stratum is a domain of polynomiality). The map  $\phi_W$  is continuous, and therefore a homeomorphism.  $\blacksquare$

## 6. $\mathcal{A}_{n,k,1}$ AS THE BOUNDED COMPLEX OF A CYCLIC HYPERPLANE ARRANGEMENT

We show that the  $m = 1$  amplituhedron  $\mathcal{B}_{n,k,1}(W)$  (or  $\mathcal{A}_{n,k,1}(Z)$ ) is homeomorphic to the complex of bounded faces of a *cyclic hyperplane arrangement* of  $n$  hyperplanes in  $\mathbb{R}^k$ . It then follows from a result of Dong [Don08] that it is homeomorphic to a ball. This story is somewhat analogous to that of  $k = 1$  amplituhedra  $\mathcal{A}_{n,1,m}$ , which are *cyclic polytopes* with  $n$  vertices in  $\mathbb{P}^m$ .<sup>5</sup> Cyclic hyperplane arrangements have been studied by Shannon [Sha79], Ziegler [Zie93], Ramírez Alfonsín [RA99], and Forge and Ramírez Alfonsín [FRA01]. For an introduction to hyperplane arrangements, see [Sta07].

**Remark 6.1.** In the literature, a *cyclic hyperplane arrangement* of  $n$  hyperplanes in  $\mathbb{R}^k$  is usually defined to be an arrangement with hyperplanes

$$(6.2) \quad \{x \in \mathbb{R}^k : t_i x_1 + t_i^2 x_2 + \cdots + t_i^k x_k + 1 = 0\} \quad (i \in [n]),$$

where  $0 < t_1 < \cdots < t_n$ . We will need to consider more general hyperplane arrangements, whose hyperplanes are of the form

$$(6.3) \quad \{x \in \mathbb{R}^k : A_{i,1}x_1 + \cdots + A_{i,k}x_k + A_{i,0} = 0\} \quad (i \in [n]),$$

such that

$$\text{colspan}((A_{i,j})_{1 \leq i \leq n, 1 \leq j \leq k}) \in \text{Gr}_{k,n}^{>0} \quad \text{and} \quad \text{colspan}((A_{i,j})_{1 \leq i \leq n, 0 \leq j \leq k}) \in \text{Gr}_{k+1,n}^{>0}.$$

(We will require that this latter subspace is  $W$ .) The hyperplane arrangement (6.2) is of this form by Vandermonde's identity. Theorem 6.16 implies that all hyperplane arrangements of the form (6.3) are isomorphic, so we will not be concerned with the distinction between (6.2) and (6.3). This is analogous to the situation for cyclic polytopes (see [Stu88]).

<sup>5</sup>In fact, the  $k = 1$  amplituhedra  $\mathcal{A}_{n,1,m}$  are precisely the *alternating polytopes* of dimension  $m$  with  $n$  vertices in  $\mathbb{P}^m$ , as follows from work of Sturmfels [Stu88]. Alternating polytopes are cyclic polytopes which have the additional property that every induced subpolytope is also cyclic. See [BLVS<sup>+</sup>99, pp. 396-397] for an example of a cyclic polytope which is not alternating.

**Definition 6.4.** An arrangement  $\{H_1, \dots, H_n\}$  of hyperplanes in  $\mathbb{R}^k$  is called *generic* if for all  $I \subseteq [n]$ , we have  $\dim(\bigcap_{i \in I} H_i) = n - |I|$  if  $|I| \leq k$ , and  $\bigcap_{i \in I} H_i = \emptyset$  if  $|I| > k$ .

**Remark 6.5.** Cyclic polytopes have many faces of each dimension, in the sense of the upper bound theorem of McMullen [McM70] and Stanley [Sta75]. An analogous property of cyclic hyperplane arrangements is that they have few simplicial faces of each dimension, in the sense of Shannon [Sha79]. Note that it does not make sense to look at the total number of faces, because the number of faces of a given dimension of a generic hyperplane arrangement depends only on its dimension and the number of hyperplanes [Buc43].

In order to define our hyperplane arrangement, we will use the following convention.

**Definition 6.6.** Suppose that  $V \in \text{Gr}_{k,n}^{>0}$  and  $w \in \mathbb{R}^n \setminus V$  such that  $V + w \in \text{Gr}_{k+1,n}^{>0}$ . Let  $U := V^\perp \cap (V + w)$ . Since  $\dim(U) = 1$ , we have  $V + w = V \oplus U$ . Let  $u \in U$  be the projection of  $w$  to  $U$ , i.e.  $w - u \in V$ . Since  $u \neq 0$  and  $u \in V^\perp$ , Theorem 3.4(ii) implies that  $\text{var}(u) \geq k$ . But since  $u$  also lies in  $V + w \in \text{Gr}_{k+1,n}^{>0}$ , Theorem 3.4(ii) implies that  $\overline{\text{var}}(u) \leq k$ . Therefore  $\text{var}(u) = \overline{\text{var}}(u) = k$ . But  $\text{var}(u) = \overline{\text{var}}(u)$  implies that the first component  $u_1$  of  $u$  is nonzero. We call  $w$  *positively oriented with respect to  $V$*  if  $u_1 > 0$ , and *negatively oriented* if  $u_1 < 0$ .

**Example 6.7.** Let  $V \in \text{Gr}_{1,3}^{>0}$  be the span of  $(1, 1, 1)$  (so  $n = 3$ ,  $k = 1$ ), and  $w := (1, 2, 4)$ , so that  $V + w \in \text{Gr}_{2,3}^{>0}$ . Then  $U := V^\perp \cap (V + w)$  is the line spanned by  $(-4, -1, 5)$ , and the projection of  $w$  to  $U$  is  $u := (-\frac{4}{3}, -\frac{1}{3}, \frac{5}{3})$ , since  $w - u = \frac{7}{3}(1, 1, 1) \in V$ . Since  $u_1 = -\frac{4}{3} < 0$ , by Definition 6.6  $w$  is negatively oriented with respect to  $V$ .  $\diamond$

Let us show that the cyclic hyperplane arrangements defined in (6.2) satisfy this positive orientation property. Although we will not need Proposition 6.8 in what follows, it will mean that our future characterization of face labels of  $\mathcal{H}^W$  in Proposition 6.14 applies to “classical” cyclic hyperplane arrangements (6.2). (This is due to the assumption in Definition 6.10 that  $w^{(0)}$  is positively oriented with respect to  $V$ .)

**Proposition 6.8.** Let  $0 < t_1 < \dots < t_n$ , and  $V \in \text{Gr}_{k,n}^{>0}$  be the span of the vectors  $(t_1^j, \dots, t_n^j)$  for  $1 \leq j \leq k$ . Then  $w = (1, \dots, 1)$  is positively oriented with respect to  $V$ .

**Proof.** We define the  $n \times (k + 1)$  matrix  $A$  with entries  $A_{i,j} := t_i^j$  ( $1 \leq i \leq n, 0 \leq j \leq k$ ). Then  $A$  is a *totally positive matrix*, i.e. all its minors are positive. Indeed, for  $I = \{i_1 < \dots < i_l\} \subseteq [n]$  and  $J = \{j_1 < \dots < j_l\} \subseteq \{0, 1, \dots, k\}$ , the classical definition of Schur functions implies that

$$\det(A_{I,J}) = s_{(j_l - l + 1, j_{l-1} - l + 2, \dots, j_1)}(t_{i_1}, \dots, t_{i_l}) \prod_{r,s \in [l], r < s} (t_{i_s} - t_{i_r}).$$

Since Schur functions are monomial-positive,  $\det(A_{I,J}) > 0$ .

Because  $V + w$  is the column span of  $A$ , and  $V$  is the span of the last  $k$  columns of  $A$ , we get that  $V \in \text{Gr}_{k,n}^{>0}$  and  $V + w \in \text{Gr}_{k+1,n}^{>0}$ . As in Definition 6.6, we let  $U := V^\perp \cap (V + w)$ , and  $u \in U$  be the projection of  $w$  to  $U$ . That is,  $w - u \in V$  and  $\text{var}(u) = \overline{\text{var}}(u) = k$ . We must show that  $u_1 > 0$ . We will use the following properties of totally positive  $n \times (k + 1)$  matrices  $A$ :

- $\text{var}(Ax) \leq \text{var}(x)$  for all  $x \in \mathbb{R}^{k+1}$  [Sch30];
- if  $\text{var}(Ax) = \text{var}(x)$ , then the first nonzero components of  $Ax$  and  $x$  have the same sign [GK50, Theorem V.5].

Since  $w - u \in V$ , we can write  $u = Ax$  for some  $x \in \mathbb{R}^{k+1}$  with  $x_1 = 1$ . And because  $\text{var}(u) = k$ , we have  $\text{var}(x) = k$ , whence the first nonzero component of  $u$  is positive.  $\blacksquare$

We will also need the following result of Rietsch.<sup>6</sup>

**Lemma 6.9** ([Rie]). *If  $W \in \text{Gr}_{k+m,n}^{>0}$ , where  $m \geq 0$ , then  $W$  contains a subspace in  $\text{Gr}_{k,n}^{>0}$ .*

We now define our hyperplane arrangement. A hyperplane arrangement  $\mathcal{H}$  partitions its ambient space in *faces*; maximal faces (equivalently, connected components of the complement of  $\mathcal{H}$ ) are called *regions*.

**Definition 6.10.** Given  $W \in \text{Gr}_{k+1,n}^{>0}$ , we use Lemma 6.9 to choose vectors  $w^{(1)}, \dots, w^{(k)} \in W$  such that  $V := \text{span}(w^{(1)}, \dots, w^{(k)}) \in \text{Gr}_{k,n}^{>0}$ . We extend  $\{w^{(1)}, \dots, w^{(k)}\}$  to a basis  $\{w^{(0)}, w^{(1)}, \dots, w^{(k)}\}$  of  $W$ ; after replacing  $w^{(0)}$  with  $-w^{(0)}$  if necessary, we assume that  $w^{(0)}$  is positively oriented with respect to  $V$  (see Definition 6.6).

We let  $\mathcal{H}^W$  be the hyperplane arrangement in  $\mathbb{R}^k$  with hyperplanes

$$H_i := \{x \in \mathbb{R}^k : w_i^{(1)}x_1 + \dots + w_i^{(k)}x_k + w_i^{(0)} = 0\} \text{ for } i \in [n].$$

Note that  $\mathcal{H}^W$  is generic by the first three sentences of the proof of [Sta07, Proposition 5.13].<sup>7</sup> Also note that  $\mathcal{H}^W$  depends not only on  $W$  but also on our choice of basis of  $W$ .

Given  $w \in W$ , we let  $\langle w \rangle \in \mathbb{P}(W)$  denote the line spanned by  $w$ . We define the maps

$$\begin{aligned} \Psi_{\mathcal{H}^W} : \mathbb{R}^k &\rightarrow \mathbb{P}(W), & x &\mapsto \langle x_1w^{(1)} + \dots + x_kw^{(k)} + w^{(0)} \rangle, \\ \psi_{\mathcal{H}^W} : \mathbb{R}^k &\rightarrow \{0, +, -\}^n, & x &\mapsto \text{sign}(x_1w^{(1)} + \dots + x_kw^{(k)} + w^{(0)}). \end{aligned}$$

Note that the faces of  $\mathcal{H}^W$  are precisely the nonempty fibers of  $\psi_{\mathcal{H}^W}$ . If  $\sigma \in \{0, +, -\}^n$  has a nonempty preimage under  $\psi_{\mathcal{H}^W}$ , we call this fiber the face of  $\mathcal{H}^W$  labeled by  $\sigma$ . When we identify faces with labels in this way, the face poset of  $\mathcal{H}^W$  is an induced subposet of the sign vectors  $\{0, +, -\}^n$  (see Definition 5.3).

Finally, we let  $B(\mathcal{H}^W)$  be the subcomplex of bounded faces of  $\mathcal{H}^W$ . We denote the set of sign vectors which label the bounded faces of  $\mathcal{H}^W$  by  $\mathcal{V}_{B(\mathcal{H}^W)}$ .

**Remark 6.11.** By Proposition 6.8, the classical cyclic hyperplane arrangement (6.2) is an example of such an  $\mathcal{H}^W$  from Definition 6.10.

<sup>6</sup>Rietsch [Rie] in fact proved that the totally nonnegative part  $\text{Fl}_n^{\geq 0}$  of the complete flag variety (as defined by Lusztig [Lus94, Section 8]) projects surjectively onto  $\text{Gr}_{k,n}^{\geq 0}$ , and that the Lusztig-Rietsch stratification of  $\text{Fl}_n^{\geq 0}$  projects onto Postnikov's stratification of  $\text{Gr}_{k,n}^{\geq 0}$ . In particular, given  $V \in \text{Gr}_{k,n}^{\geq 0}$ , there exists a complete flag  $V_0 \subset V_1 \subset \dots \subset V_n$  in the totally positive part  $\text{Fl}_n^{>0}$  of  $\text{Fl}_n^{\geq 0}$  with  $V_k = V$ . This immediately implies Lemma 6.9, because if  $V_0 \subset V_1 \subset \dots \subset V_n \in \text{Fl}_n^{>0}$  then  $V_j \in \text{Gr}_{j,n}^{>0}$  for  $0 \leq j \leq n$ . (See [TW15, Corollary 7.2] for a related result.) An alternative proof of Rietsch's result was given by Talaska and Williams [TW13, Theorem 6.6], by relating Postnikov's parameterizations of cells of  $\text{Gr}_{k,n}^{\geq 0}$  [Pos, Theorem 6.5] to Marsh and Rietsch's parameterizations of the Lusztig-Rietsch cells [MR04]. A direct proof of Lemma 6.9 using similar tools was given in [Lam16b, Lemma 15.6].

<sup>7</sup>We thank Richard Stanley for pointing out this argument to us.

We will show in Theorem 6.16 that  $\Psi_{\mathcal{H}^W}$  gives a homeomorphism from  $B(\mathcal{H}^W)$  to  $\mathcal{B}_{n,k,1}(W)$ . The key to the proof is establishing that  $\mathcal{V}_{B(\mathcal{H}^W)} = \overline{\text{Sign}_{n,k,1}}$ , which we do in Proposition 6.14. (Recall that  $\overline{\text{Sign}_{n,k,1}}$  is the set of nonzero sign vectors  $\sigma \in \{0, +, -\}^n$  such that  $\overline{\text{var}}(\sigma) = k$ , and if  $i \in [n]$  indexes the first nonzero component of  $\sigma$ , then  $\sigma_i = (-1)^{i-1}$ . See Definition 5.1.)

In what follows, we fix  $W \in \text{Gr}_{k+1,n}^{>0}$ , as well as a basis  $\{w^{(0)}, w^{(1)}, \dots, w^{(k)}\}$  of  $W$  and a corresponding hyperplane arrangement  $\mathcal{H}^W$ , as in Definition 6.10.

**Lemma 6.12.**

- (i) If  $\sigma \in \{+, -\}^n$  satisfies  $\text{var}(\sigma) \leq k - 1$ , then  $\sigma$  labels an unbounded region of  $\mathcal{H}^W$ .
- (ii) If  $\sigma \in \{0, +, -\}^n$  labels an unbounded face of  $\mathcal{H}^W$ , then  $\overline{\text{var}}(\sigma) \leq k - 1$ .

**Proof.** (i) Suppose that  $\sigma \in \{+, -\}^n$  with  $\text{var}(\sigma) \leq k - 1$ . Since  $\text{span}(w^{(1)}, \dots, w^{(k)}) \in \text{Gr}_{k,n}^{>0}$ , Lemma 3.7(i) implies that we can write  $\sigma = \text{sign}(w)$  for some  $w \in \text{span}(w^{(1)}, \dots, w^{(k)})$ . Then for all  $t > 0$  sufficiently large we have  $\text{sign}(tw + w^{(0)}) = \sigma$ , whence  $\psi_{\mathcal{H}^W}^{-1}(\sigma)$  is unbounded.

(ii) Given  $\sigma \in \{0, +, -\}^n$  such that  $\psi_{\mathcal{H}^W}^{-1}(\sigma)$  is an unbounded face of  $\mathcal{H}^W$ , take a sequence  $(x^{(j)})_{j \in \mathbb{N}}$  in  $\psi_{\mathcal{H}^W}^{-1}(\sigma)$  with  $\lim_{j \rightarrow \infty} \|x^{(j)}\| = \infty$ .

*Claim.*  $\lim_{j \rightarrow \infty} \|x_1^{(j)}w^{(1)} + \dots + x_k^{(j)}w^{(k)} + w^{(0)}\| = \infty$ .

*Proof of Claim.* Let  $c \in \mathbb{R}$  be the minimum value of  $\|x_1w^{(1)} + \dots + x_kw^{(k)}\|$  on the compact set  $\{x \in \mathbb{R}^k : \|x\| = 1\}$ . Since  $w^{(1)}, \dots, w^{(k)}$  are linearly independent, we have  $c > 0$ . Hence by the triangle inequality,

$$\|x_1^{(j)}w^{(1)} + \dots + x_k^{(j)}w^{(k)} + w^{(0)}\| \geq \|x_1^{(j)}w^{(1)} + \dots + x_k^{(j)}w^{(k)}\| - \|w^{(0)}\| \geq c\|x^{(j)}\| - \|w^{(0)}\|$$

for  $j \in \mathbb{N}$ . The claim follows when we send  $j$  to  $\infty$ .  $\square$

By the claim, the following sequence is well defined for  $j$  sufficiently large:

$$\left( \frac{x_1^{(j)}w^{(1)} + \dots + x_k^{(j)}w^{(k)} + w^{(0)}}{\|x_1^{(j)}w^{(1)} + \dots + x_k^{(j)}w^{(k)} + w^{(0)}\|} \right)_{j \in \mathbb{N}}.$$

Since  $\{x \in \mathbb{R}^k : \|x\| = 1\}$  is compact, this sequence has a convergent subsequence; we let  $v \in \mathbb{R}^n$  denote one of its limit points. Then since  $\lim_{j \rightarrow \infty} \|x^{(j)}\| = \infty$ , we have  $v \in \text{span}(w^{(1)}, \dots, w^{(k)}) \setminus \{0\}$ . We get  $\overline{\text{var}}(\sigma) \leq \overline{\text{var}}(v) \leq k - 1$ , where the first inequality holds since  $\text{sign}(v)$  is obtained from  $\text{sign}(\sigma)$  by possibly setting some nonzero components to zero, and the second inequality follows from Theorem 3.4(ii).  $\blacksquare$

**Lemma 6.13.** *The map  $\mathcal{V}_{B(\mathcal{H}^W)} \rightarrow \overline{\mathbb{P}\text{Sign}_{n,k,1}}$ ,  $\sigma \mapsto \sigma$  is a poset isomorphism. In other words, the face poset of the bounded faces of  $\mathcal{H}^W$  is isomorphic to the face poset of  $\mathcal{B}_{n,k,1}(W)$ .*

**Proof.** First let us show that the map  $\mathcal{V}_{B(\mathcal{H}^W)} \rightarrow \overline{\mathbb{P}\text{Sign}_{n,k,1}}$  is well defined, i.e. given  $\sigma \in \mathcal{V}_{B(\mathcal{H}^W)}$ , we have  $\overline{\text{var}}(\sigma) = k$ . Since  $B(\mathcal{H}^W)$  equals the closure of the union of the bounded regions of  $\mathcal{H}^W$  [Don08] (see also [Sta07, Chapter 1, Exercise 7(d)]), the face labeled by  $\sigma$  is contained in the closure of some bounded region of  $\mathcal{H}^W$ , labeled by, say,  $\tau \in \{+, -\}^n$ . Therefore  $\tau \geq \sigma$ , where the partial order is the one on  $\{0, +, -\}^n$  from Definition 5.3. By

Lemma 6.12(i) we have  $\text{var}(\tau) \geq k$ , and so  $\overline{\text{var}}(\sigma) \geq \text{var}(\tau) \geq k$ . But by Theorem 3.4(ii) we also have  $\overline{\text{var}}(\sigma) \leq k$ , so  $\overline{\text{var}}(\sigma) = k$ , as desired.

Therefore the map  $\mathcal{V}_{B(\mathcal{H}^W)} \rightarrow \overline{\text{PSign}_{n,k,1}}$  is a poset homomorphism. To see that it is surjective, note that if  $\sigma \in \{0, +, -\}^n \setminus \{0\}$  satisfies  $\overline{\text{var}}(\sigma) = k$ , then  $\sigma = \text{sign}(w)$  for some  $w \in W$  by Lemma 3.7(i). When we write  $w$  in terms of the basis  $w^{(0)}, w^{(1)}, \dots, w^{(k)}$ , the coefficient of  $w^{(0)}$  is nonzero. (Otherwise  $w \in \text{span}(w^{(1)}, \dots, w^{(k)}) \in \text{Gr}_{k,n}^{>0}$ , implying  $\overline{\text{var}}(w) \leq k-1$  by Theorem 3.4(ii).) Rescaling  $w$  by a positive real number so that this coefficient is  $\pm 1$ , we see that we can write

$$w = x_1 w^{(1)} + \dots + x_k w^{(k)} \pm w^{(0)}$$

for some  $x \in \mathbb{R}^k$ . Therefore either  $\sigma$  or  $-\sigma$  labels a face of  $\mathcal{H}^W$ , and such a face is bounded by Lemma 6.12(ii).

It remains to show that the map  $\mathcal{V}_{B(\mathcal{H}^W)} \rightarrow \overline{\text{PSign}_{n,k,1}}$  is injective and that its inverse is a poset homomorphism. It suffices to prove that there do not exist  $\sigma, \tau \in \{0, +, -\}^n$  labeling bounded faces of  $\mathcal{H}^W$  such that  $\sigma \leq -\tau$ . Suppose otherwise that there exist such  $\sigma, \tau$ . Take  $x, y \in \mathbb{R}^k$  with  $\psi_{\mathcal{H}^W}(x) = \sigma$  and  $\psi_{\mathcal{H}^W}(y) = \tau$ . Subtracting, we get  $\text{sign}((x_1 - y_1)w^{(1)} + \dots + (x_k - y_k)w^{(k)}) = -\tau$ . Since  $\text{span}(w^{(1)}, \dots, w^{(k)}) \in \text{Gr}_{k,n}^{>0}$ , Theorem 3.4(ii) implies that  $\overline{\text{var}}(\tau) \leq k-1$ . But we showed in the first paragraph that  $\overline{\text{var}}(\tau) = k$ .  $\blacksquare$

**Proposition 6.14** (The face labels of  $\mathcal{H}^W$ ).

- (i) The labels of the bounded faces of  $\mathcal{H}^W$  are precisely  $\overline{\text{Sign}_{n,k,1}}$ , i.e.  $\mathcal{V}_{B(\mathcal{H}^W)} = \overline{\text{Sign}_{n,k,1}}$ .
- (ii) The labels of the unbounded faces of  $\mathcal{H}^W$  are precisely  $\sigma \in \{0, +, -\}^n$  with  $\overline{\text{var}}(\sigma) \leq k-1$ .

**Example 6.15.** Let  $n := 5, k := 2, m := 1$ , and  $W \in \text{Gr}_{3,5}^{>0}$  have the basis

$$w^{(0)} := (-1, -1, -1, -1, -1), \quad w^{(1)} := (0, 1, 2, 3, 4), \quad w^{(2)} := (10, 6, 3, 1, 0).$$

Note that  $V := \text{span}(w^{(1)}, w^{(2)}) \in \text{Gr}_{2,5}^{>0}$ . The projection  $u$  of  $w^{(0)}$  to  $V^\perp \cap W$  is

$$u := w^{(0)} + \frac{1}{831}(232w^{(1)} + 90w^{(2)}) = \frac{1}{831}(69, -59, -97, -45, 97).$$

Since  $u_1 > 0$ , by Definition 6.6  $w^{(0)}$  is positively oriented with respect to  $V$ .

The hyperplane arrangement  $\mathcal{H}^W$  from Definition 6.10 consists of 5 lines in  $\mathbb{R}^2$ :

$$\ell_1: 10y = 1, \quad \ell_2: x + 6y = 1, \quad \ell_3: 2x + 3y = 1, \quad \ell_4: 3x + y = 1, \quad \ell_5: 4x = 1.$$

See Figure 10 and Figure 11 for  $B(\mathcal{H}^W)$  labeled by sign vectors and J-diagrams, respectively. (The positive side of each line is above and to the right of it.) By Proposition 6.14, the bounded faces of  $\mathcal{H}^W$  are labeled by sign vectors in  $\overline{\text{Sign}_{5,2,1}}$ , and the bounded regions by sign vectors in  $\text{Sign}_{5,2,1}$ . The unbounded regions are labeled by the sign vectors  $\sigma \in \{0, +, -\}^5$  satisfying  $\overline{\text{var}}(\sigma) \leq 1$ . By Theorem 6.16, we have  $\mathcal{B}_{5,2,1}(W) \cong B(\mathcal{H}^W)$ .  $\diamond$

**Proof** (of Proposition 6.14). (i) By Lemma 6.13 and Lemma 5.4(ii),  $\mathcal{V}_{B(\mathcal{H}^W)}$  equals either  $\overline{\text{Sign}_{n,k,1}}$  or  $-\overline{\text{Sign}_{n,k,1}}$ . We must rule out the latter possibility. Recall that by construction,  $w^{(0)}$  is positively oriented with respect to  $V := \text{span}(w^{(1)}, \dots, w^{(k)})$ . According to Definition 6.6, there exist  $x_1, \dots, x_k \in \mathbb{R}$  such that  $u := x_1 w^{(1)} + \dots + x_k w^{(k)} + w^{(0)}$  satisfies  $\text{var}(u) = \overline{\text{var}}(u) = k$  and  $u_1 > 0$ . Let  $\sigma := \text{sign}(u)$ . Then  $\sigma$  labels a face of  $\mathcal{H}^W$ ,



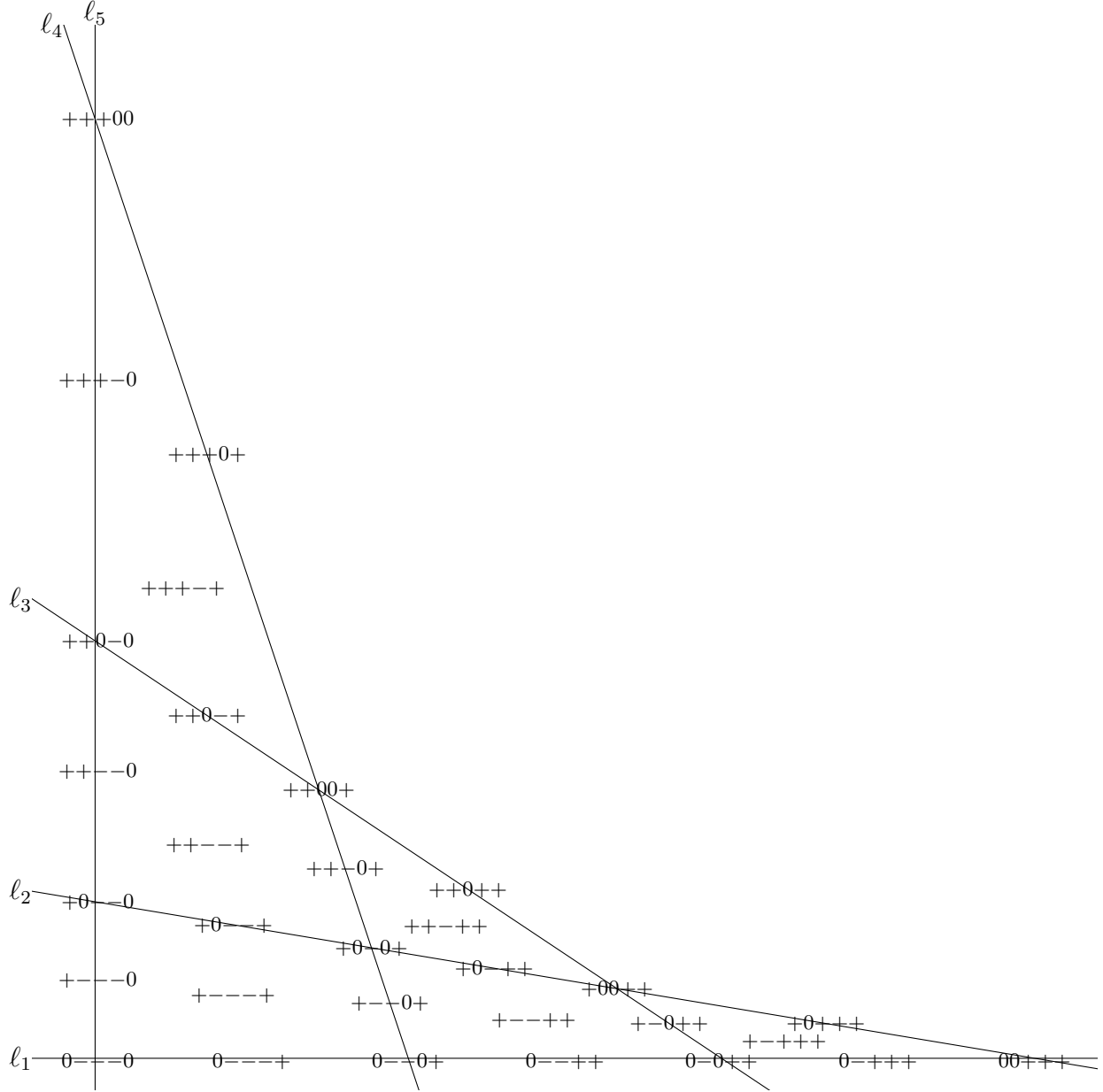


FIGURE 10. The hyperplane arrangement  $\mathcal{H}^W$  from Example 6.15, with  $\mathcal{B}_{5,2,1}(W) \cong B(\mathcal{H}^W)$ . Its bounded faces are labeled by sign vectors.

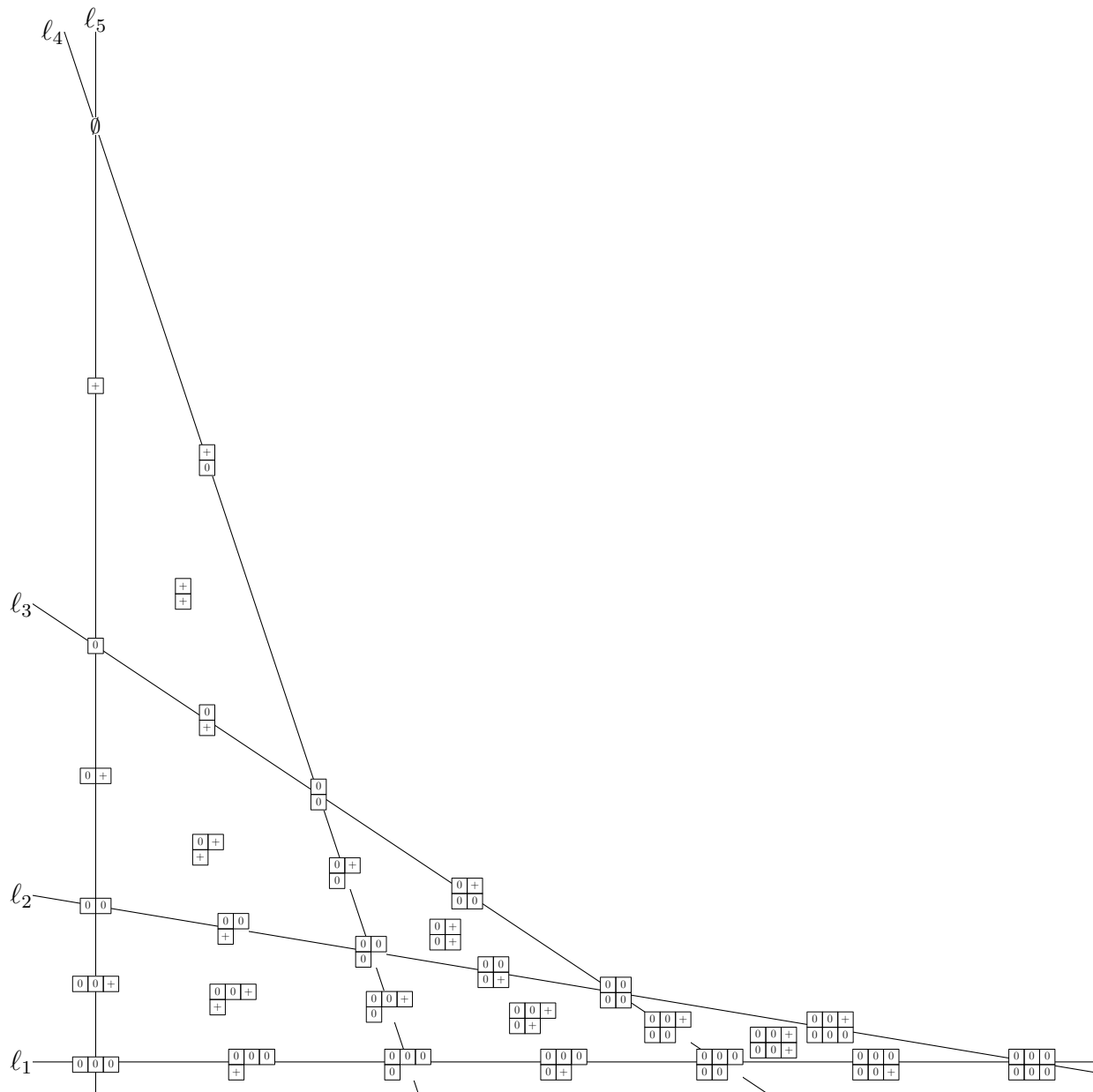


FIGURE 11. The hyperplane arrangement  $\mathcal{H}^W$  from Example 6.15, with  $\mathcal{B}_{5,2,1}(W) \cong B(\mathcal{H}^W)$ . Its bounded faces are labeled by  $\mathbb{J}$ -diagrams.

which is a bounded face by Lemma 6.12(ii). We get  $\sigma \in \mathcal{V}_{B(\mathcal{H}^W)}$ , and  $\sigma_1 = +$  implies  $\mathcal{V}_{B(\mathcal{H}^W)} \neq -\overline{\text{Sign}}_{n,k,1}$ .

(ii) One direction follows from Lemma 6.12(ii). For the other direction, we will use the following fact about generic hyperplane arrangements  $\mathcal{H}$  in  $\mathbb{R}^k$ : if  $\tau$  labels a face of  $\mathcal{H}$ , then  $\sigma$  also labels a face of  $\mathcal{H}$  for all  $\sigma \geq \tau$ . (This follows from the fact that the normal vectors of any  $k$  or fewer hyperplanes are linearly independent.)

Given  $\sigma \in \{0, +, -\}^n$  with  $\overline{\text{var}}(\sigma) \leq k - 1$ , we must show that  $\sigma$  labels a face of  $\mathcal{H}^W$  (whence this face is unbounded by part (i)). Since  $\mathcal{H}^W$  is generic, it suffices to construct  $\tau \leq \sigma$  which labels a face of  $\mathcal{H}^W$ . Our strategy will be to modify the sign vector  $\sigma$  until we get a sign vector  $\tau \leq \sigma$  with  $\tau \in \overline{\text{Sign}}_{n,k,1}$ , which then implies that  $\tau$  labels a bounded face of  $\mathcal{H}^W$  by part (i).

Set  $\sigma' := \text{alt}(\sigma)$ . By Lemma 3.3(i), we have  $\text{var}(\sigma') \geq n - k$ . In particular, the fact that  $k < n$  implies that  $\sigma'$  has a positive component. Take  $i \in [n]$  minimum with  $\sigma'_i = +$ , and let  $\sigma''$  be obtained from  $\sigma'$  by setting to zero all components  $j$  with  $j < i$ . Note that  $\text{var}(\sigma'') \geq \text{var}(\sigma') - 1 \geq n - k - 1$ . Now we repeatedly set the last nonzero component of  $\sigma''$  to zero, until we obtain a sign vector  $\tau'$  with  $\text{var}(\tau') = n - k - 1$ . Letting  $\tau := \text{alt}(\tau')$ , we have  $\tau \leq \sigma$ , and  $\overline{\text{var}}(\tau) = k$  by Lemma 3.3(i). Since the first nonzero component of  $\tau'$  equals  $+$ , we have  $\tau \in \overline{\text{Sign}}_{n,k,1}$ , as desired. ■

We are now ready to show that  $\mathcal{B}_{n,k,1}(W) \cong B(\mathcal{H}^W)$ .

**Theorem 6.16.** *The restriction of  $\Psi_{\mathcal{H}^W}$  to  $B(\mathcal{H}^W)$  is a homeomorphism from  $B(\mathcal{H}^W)$  to  $\mathcal{B}_{n,k,1}(W)$ , and induces an isomorphism of posets on the strata of  $B(\mathcal{H}^W)$  and  $\mathcal{B}_{n,k,1}(W)$ . Explicitly,  $\Psi_{\mathcal{H}^W}$  sends the stratum  $\psi_{\mathcal{H}^W}^{-1}(\sigma)$  of  $B(\mathcal{H}^W)$  to the stratum  $\mathcal{B}_\sigma(W)$  of  $\mathcal{B}_{n,k,1}(W)$ , for all  $\sigma \in \overline{\text{Sign}}_{n,k,1}$ .*

**Proof.** To see that  $\Psi_{\mathcal{H}^W}$  is injective, note that if  $\Psi_{\mathcal{H}^W}(x) = \Psi_{\mathcal{H}^W}(y)$  for some  $x, y \in \mathbb{R}^k$ , then there exists  $t \in \mathbb{R} \setminus \{0\}$  such that  $x_1 w^{(1)} + \cdots + x_k w^{(k)} + w^{(0)} = t(y_1 w^{(1)} + \cdots + y_k w^{(k)} + w^{(0)})$ . The linear independence of the vectors  $w^{(0)}, w^{(1)}, \dots, w^{(k)}$  implies that  $t = 1$  and  $x_i = y_i$  for all  $i \in [k]$ , so  $x = y$ .

Recall that by Corollary 3.16,  $\mathcal{B}_{n,k,1}(W) = \{w \in \mathbb{P}(W) : \overline{\text{var}}(w) = k\}$ . Let us show that  $\Psi_{\mathcal{H}^W}(\mathbb{R}^k)$  contains  $\mathcal{B}_{n,k,1}(W)$ . It suffices to observe that when we express an element of  $\mathcal{B}_{n,k,1}(W)$  as a linear combination of  $w^{(0)}, w^{(1)}, \dots, w^{(k)}$ , the coefficient of  $w^{(0)}$  is nonzero. This follows from Theorem 3.4(ii), and the fact that  $\text{span}(w^{(1)}, \dots, w^{(k)}) \in \text{Gr}_{k,n}^{>0}$ .

Now we let  $\mathcal{Q} := \Psi_{\mathcal{H}^W}^{-1}(\mathcal{B}_{n,k,1}(W))$ . We have shown that  $\Psi_{\mathcal{H}^W}$  is a homeomorphism from  $\mathcal{Q}$  to  $\mathcal{B}_{n,k,1}$ . Recall that  $\mathcal{B}_{n,k,1}(W)$  is stratified by the sign vectors in  $\overline{\text{PSign}}_{n,k,1}(W)$ . Therefore  $\mathcal{Q}$  is the union of the faces of  $\mathcal{H}^W$  labeled by  $\sigma \in \{0, +, -\}^n$  satisfying  $\overline{\text{var}}(\sigma) = k$ . By Proposition 6.14, this is precisely  $B(\mathcal{H}^W)$ . The fact that  $\Psi_{\mathcal{H}^W}$  induces a poset isomorphism on strata follows from Lemma 6.13. ■

**Remark 6.17.** It follows from Theorem 6.16 that the amplituhedron  $\mathcal{B}_{n,k,1}$  is a regular cell complex, and in particular its strata  $\mathcal{B}_\sigma(W)$  are homeomorphic to open balls. Using the results of Section 5, we can also index the cells of the amplituhedron by  $\mathbf{J}$ -diagrams and matroids, in which case we will use the notation  $\mathcal{B}_D(W)$  and  $\mathcal{B}_M(W)$ , respectively.

**Corollary 6.18.** *The amplituhedron  $\mathcal{B}_{n,k,1}(W)$  (and also  $\mathcal{A}_{n,k,1}(Z)$ ) is homeomorphic to a ball of dimension  $k$ .*

**Proof.** This follows from Theorem 6.16 together with Dong's result [Don08, Theorem 3.1] that the bounded complex of a uniform affine oriented matroid (of which the bounded complex of a generic hyperplane arrangement is a special case) is a piecewise linear ball. ■

Recall from Theorem 5.17 that the amplituhedron  $\mathcal{B}_{n,k,1}(W)$  is homeomorphic to a subcomplex of  $\text{Gr}_{k,n}^{\geq 0}$ ; indeed,  $\mathcal{B}_{n,k,1}(W)$  inherits a cell decomposition whose face poset is an induced subposet of the face poset of  $\text{Gr}_{k,n}^{\geq 0}$ . Another way to prove Corollary 6.18 would be to show that the face poset of  $\mathcal{B}_{n,k,1}(W)$ , with a new top element  $\hat{1}$  adjoined, is *shellable*. Then since the cell decomposition of  $\mathcal{B}_{n,k,1}(W)$  is regular, and its face poset is *pure* and *sub-thin* (this follows from Theorem 6.16), a result of Björner [Bjö84, Proposition 4.3(c)] would imply that  $\mathcal{B}_{n,k,1}(W)$  is homeomorphic to a ball.

**Problem 6.19.** *Show that the face poset of  $\mathcal{B}_{n,k,1}(W)$  with a top element  $\hat{1}$  adjoined is shellable, e.g. by finding an EL-labeling.*

**Remark 6.20.** Note that in earlier work the second author proved that the face poset of  $\text{Gr}_{k,n}^{\geq 0}$  is thin and shellable [Wil07], which shows that the same is true for any induced subposet. However, this does not solve Problem 6.19, because after adjoining  $\hat{1}$ , the face poset of  $\mathcal{B}_{n,k,1}(W)$  is no longer an induced subposet of the face poset of  $\text{Gr}_{k,n}^{\geq 0}$ .

As a further corollary of Theorem 6.16, we obtain the generating function for the stratification of  $\mathcal{B}_{n,k,1}(W)$  with respect to dimension, since Buck found the corresponding generating function of  $B(\mathcal{H})$  for a generic hyperplane arrangement  $\mathcal{H}$  (which only depends on its dimension and the number of hyperplanes).

**Corollary 6.21** ([Buc43]). *Let  $f_{n,k,1}(q) := \sum_{\text{strata } S \text{ of } \mathcal{B}_{n,k,1}(W)} q^{\dim(S)} \in \mathbb{N}[q]$  be the generating function for the stratification of  $\mathcal{B}_{n,k,1}(W)$ , with respect to dimension. Then*

$$f_{n,k,1}(q) = \sum_{i=0}^k \binom{n-k-1+i}{i} \binom{n}{k-i} q^i = \sum_{j=0}^k \binom{n-k-1+j}{j} (1+q)^j.$$

For example, we have  $f_{5,3,1}(q) = 4q^3 + 15q^2 + 20q + 10$ , which we invite the reader to verify from Figure 1.

**Proof.** By the corollary to Theorem 3 of [Buc43], for  $i \in \mathbb{N}$  the coefficient of  $q^i$  in  $f_{n,k,1}(q)$  equals  $\frac{k+1}{n-k+i} \binom{k}{i} \binom{n}{k+1}$ , which we can rewrite as  $\binom{n-k-1+i}{i} \binom{n}{k-i}$ . This gives the first sum above. For the last equality above, note that the coefficient of  $q^i$  in  $\sum_{j=0}^k \binom{n-k-1+j}{j} (1+q)^j$  equals

$$\begin{aligned} \sum_{j=i}^k \binom{n-k-1+j}{j} \binom{j}{i} &= \binom{n-k-1+i}{i} \sum_{j=i}^k \binom{n-k-1+j}{n-k-1+i} \\ &= \binom{n-k-1+i}{i} \binom{n}{n-k+i} \end{aligned}$$

by the hockey-stick identity. ■

**Remark 6.22.** By substituting  $q = -1$  into the last expression in Corollary 6.21, it is easy to check that the Euler characteristic of  $\mathcal{B}_{n,k,1}(W)$  equals 1.

## 7. HOW CELLS OF $\mathcal{A}_{n,k,1}$ FIT TOGETHER

In this section we will address how cells of the  $m = 1$  amplituhedron fit together. In particular, we will explicitly work out when two maximal cells are adjacent, and which cells lie in the boundary of  $\mathcal{B}_{n,k,1}(W)$ , in terms of  $\mathbb{J}$ -diagrams, sign vectors, positroids, and decorated permutations. See Figure 10 and Figure 11 for examples when  $n = 5$  and  $k = 2$ .

**Proposition 7.1** (Adjacency of maximal cells in the  $m = 1$  amplituhedron).

Given  $D_1, D_2 \in \mathcal{D}_{n,k,1}$ , the following are equivalent:

- (i) the cells  $\mathcal{B}_{D_1}(W)$  and  $\mathcal{B}_{D_2}(W)$  in  $\mathcal{B}_{n,k,1}(W)$  are adjacent, i.e. their closures intersect in a cell  $\mathcal{B}_{D'}(W)$  of codimension 1, where  $D' \in \overline{\mathcal{D}_{n,k,1}}$  is necessarily unique;
- (ii) the Young diagrams of  $D_1$  and  $D_2$  differ by a single box, in which case we obtain  $D'$  from either  $D_1$  or  $D_2$  by including this box with a 0 inside it;
- (iii) there exists  $2 \leq i \leq n-1$  such that the sign vectors  $\sigma(D_1)$  and  $\sigma(D_2)$  differ precisely in component  $i$ , in which case we obtain  $\sigma(D')$  from either  $\sigma(D_1)$  or  $\sigma(D_2)$  by setting component  $i$  to 0;
- (iv) there exists  $2 \leq i \leq n-1$  such that we can obtain the partition of  $[n]$  for  $M(D_1)$  (in the sense of Definition 5.9) from the partition of  $[n]$  for  $M(D_2)$  by moving  $i$  from one interval to another, in which case we obtain  $M(D')$  from either  $M(D_1)$  or  $M(D_2)$  by turning  $i$  into a coloop;
- (v) there exists  $2 \leq i \leq n-1$  such that  $\pi(D_2) = s_{i-1}\pi(D_1)s_i$ , in which case  $\pi(D')$  equals either  $\pi(D_1)s_i = s_{i-1}\pi(D_2)$  or  $s_{i-1}\pi(D_1) = \pi(D_2)s_i$ , whichever has exactly  $k-1$  inversions. (Here  $s_j$  denotes the simple transposition exchanging  $j$  and  $j+1$ , and all fixed points are colored black.)

**Proof.** The uniqueness of  $D'$  and the equivalence (i)  $\Leftrightarrow$  (iii) follows from the fact that  $\mathcal{B}_{n,k,1}(W) \cong B(\mathcal{H}^W)$  (Theorem 6.16); note that since  $\sigma(D_1), \sigma(D_2) \in \text{Sign}_{n,k,1}$ , the sign vectors cannot differ in their first or last component. The equivalence (ii)  $\Leftrightarrow$  (iii) follows from Lemma 5.8, and (ii)  $\Leftrightarrow$  (iv) follows from the bijection in Definition 5.12. The equivalence (ii)  $\Leftrightarrow$  (v) follows from the bijection in Lemma 4.2, using the following explicit description of  $\pi' := s_{i-1}\pi s_i$  for any  $m = 1$  BCFW permutation  $\pi$ :

- if  $i$  is the minimum value in its cycle and not a fixed point, then we obtain  $\pi'$  from  $\pi$  in cycle notation by moving  $i$  to the cycle with  $i-1$ , to the left of  $i-1$ ;
- if  $i$  is the maximum value in its cycle and not a fixed point, then we obtain  $\pi'$  from  $\pi$  in cycle notation by moving  $i$  to the cycle with  $i+1$ , to the right of  $i+1$ ;
- if  $\pi(i) = i$ , then  $\pi'$  has  $k+1$  anti-excedances (and hence does not index a cell of  $\text{Gr}_{k,n}^{\geq 0}$ , by Lemma 2.6);
- if  $i-1$ ,  $i$ , and  $i+1$  are all in the same cycle, then  $\pi' = \pi$ . ■

Since our stratification of the amplituhedron  $\mathcal{B}_{n,k,1}(W)$  is a regular cell decomposition of a ball (see Remark 6.17 and Corollary 6.18), it is interesting to characterize which cells (necessarily of codimension at least 1) comprise its boundary. Note that by our identification

of cell complexes  $\mathcal{B}_{n,k,1}(W) \cong B(\mathcal{H}^W)$ , every cell of  $\mathcal{B}_{n,k,1}(W)$  lies in either the interior or the boundary of  $\mathcal{B}_{n,k,1}(W)$ .

**Proposition 7.2** (Boundary of the  $m = 1$  amplituhedron).

Given  $D \in \overline{\mathcal{D}}_{n,k,1}$ , the following are equivalent:

- (i) the cell  $\mathcal{B}_D(W)$  of  $\mathcal{B}_{n,k,1}(W)$  is contained in the interior;
- (ii)  $D$  has  $k$  (nonempty) rows, and for all  $r \in [k]$  such that row  $r$  of  $D$  has no  $+$ 's, row  $r - 1$  of  $D$  is longer than row  $r$  (where row 0 has length  $n - k$ );
- (iii)  $\text{var}(\sigma(D)) = k$ ;
- (iv)  $C \subseteq \{\min(E_1), \dots, \min(E_{n-k})\} \setminus \{1\}$ , where  $(E_1 \sqcup \dots \sqcup E_{n-k}, C)$  corresponds to  $M(D)$  as in Lemma 5.10;
- (v) if  $i \in [n]$  is a white fixed point of  $\pi$ , then  $2 \leq i \leq n - 1$  and  $i - 1$  is not an anti-excedance of  $\pi$ .

We remark that the result applies even when  $D$  is in  $\mathcal{D}_{n,k,1}$  (i.e. the corresponding cell has full dimension), in which case all of the above properties hold.

**Proof.** We fix a hyperplane arrangement  $\mathcal{H}^W$  as in Definition 6.10, so that  $\mathcal{B}_{n,k,1}(W) \cong B(\mathcal{H}^W)$  by Theorem 6.16, and we let  $\sigma$  denote  $\sigma(D)$ .

(i)  $\Rightarrow$  (iii): Suppose that  $\text{var}(\sigma) \neq k$ . Then  $\text{var}(\sigma) < k$ , and we can construct  $\tau \in \{+, -\}^n$  with  $\tau \geq \sigma$  and  $\text{var}(\tau) = \text{var}(\sigma)$ . For example, do the following repeatedly: take  $i \in [n]$  such that component  $i$  is zero but either component  $i - 1$  or  $i + 1$  is nonzero, and make component  $i$  nonzero and equal to either component  $i - 1$  or  $i + 1$ . Then  $\tau$  labels an unbounded face of  $\mathcal{H}^W$  by Proposition 6.14(ii), whose closure contains the face labeled by  $\sigma$ .

(iii)  $\Rightarrow$  (i): The faces of  $\mathcal{H}^W$  whose closure contains the face labeled by  $\sigma$  are labeled by  $\tau \in \{+, -\}^n$  with  $\tau \geq \sigma$ . If  $\text{var}(\sigma) = k$  then  $\overline{\text{var}}(\tau) \geq \text{var}(\tau) \geq \text{var}(\sigma) = k$ , whence the face labeled by  $\tau$  is bounded by Proposition 6.14(ii).

(ii)  $\Leftrightarrow$  (iii): Observe that  $\text{var}(\sigma) = \overline{\text{var}}(\sigma)$  if and only if for all  $i \in [n]$  such that  $\sigma_i = 0$ , we have that  $i \neq 1, n$ , and that  $\sigma_{i-1}, \sigma_{i+1}$  are nonzero and of opposite sign. This condition is equivalent to (ii) by Lemma 5.8.

(ii)  $\Leftrightarrow$  (iv): This follows from the bijection in Definition 5.12.

(ii)  $\Leftrightarrow$  (v): This follows from the bijection in Lemma 2.6. ■

## 8. THE IMAGE IN $\mathcal{A}_{n,k,1}$ OF AN ARBITRARY CELL OF $\text{Gr}_{k,n}^{\geq 0}$

In this section we study the image in the  $m = 1$  amplituhedron of an arbitrary cell of the totally nonnegative Grassmannian. In particular, we describe the image of an arbitrary cell in terms of strata of  $\mathcal{B}_{n,k,1}(W)$  (Lemma 8.2), we compute the dimension of the image of an arbitrary cell (Proposition 8.4), and we characterize the cells which map injectively to the  $m = 1$  amplituhedron (Theorem 8.10). Since we have a regular cell decomposition of the amplituhedron  $\mathcal{B}_{n,k,1}(W)$  using the  $m = 1$  BCFW cells and their closures (which can be indexed by the J-diagrams in  $\overline{\mathcal{D}}_{n,k,1}$ ), it is also natural to ask how to describe the image of an arbitrary cell of  $\text{Gr}_{k,n}^{\geq 0}$  in terms of  $\overline{\mathcal{D}}_{n,k,1}$ . We answer this question (Theorem 8.10) for cells which map injectively into the amplituhedron.

Let us fix a subspace  $W \in \text{Gr}_{k+1,n}^{\geq 0}$  for the remainder of the section. Given a J-diagram  $D$  inside a  $k \times (n - k)$  rectangle, let  $S_D$  denote its corresponding positroid cell, i.e. the cell

$S_{M(D)}$  from Definition 2.1, where  $M(D)$  is the positroid corresponding to  $D$  from Section 2. Recall from Remark 6.17 that

$$\mathcal{B}_D(W) := \{V^\perp \cap W : V \in S_D\}$$

is the image of  $S_D$  in  $\mathcal{B}_{n,k,1}(W)$ . (It is equivalent to study the image  $\tilde{Z}(S_D)$  in  $\mathcal{A}_{n,k,1}(Z)$  by Proposition 3.12, where  $Z$  is any  $(k+1) \times n$  matrix whose rows span  $W$ , but we will find it more convenient to work in  $\mathcal{B}_{n,k,1}(W)$ .)

**Definition 8.1.** Let  $D$  be a J-diagram of type  $(k, n)$ . Fix  $V \in S_D$  and define

$$\mathcal{V}(D) := \{\text{sign}(v) : v \in V^\perp\} \subseteq \{0, +, -\}^n.$$

In terms of oriented matroids [BLVS<sup>+</sup>99],  $\mathcal{V}(D)$  is the set of *vectors* of the *positive orientation* of  $M(D)$ , and so does not depend on our choice of  $V \in S_D$ .

A basic observation is that  $\mathcal{B}_D(W)$  depends precisely on the sign vectors in  $\mathcal{V}(D)$  which minimize  $\overline{\text{var}}(\cdot)$ .

**Lemma 8.2.** For  $D$  a J-diagram of type  $(k, n)$ , we have

$$\mathcal{B}_D(W) = \bigcup \{\mathcal{B}_\sigma(W) : \sigma \in \mathcal{V}(D) \text{ with } \overline{\text{var}}(\sigma) = k\}.$$

Recall that  $\mathcal{B}_\sigma(W)$  is the  $\sigma$ -stratum of  $\mathcal{B}_{n,k,1}(W)$  from Definition 5.2. While Lemma 8.2 is not very explicit (it requires being able to compute the sign vectors in  $\mathcal{V}(D)$ ), we will give a more concrete description of the images of certain cells in Theorem 8.10.

**Proof.** By Corollary 3.16, the left-hand side is contained in the right-hand side. Conversely, given  $\sigma \in \mathcal{V}(D)$  with  $\overline{\text{var}}(\sigma) = k$  and an element  $\text{span}(w) \in \mathcal{B}_\sigma(W)$  (where  $w \in W \setminus \{0\}$ ), let us show that  $\text{span}(w) \in \mathcal{B}_D(W)$ . Take any  $V \in S_D$ . Since  $\sigma \in \mathcal{V}(D)$ , there exists  $v \in V^\perp$  with  $\text{sign}(v) = \sigma$ . Since  $\text{sign}(v) = \text{sign}(w)$ , there exist  $c_1, \dots, c_n > 0$  such that  $(c_1 v_1, \dots, c_n v_n) = w$ . We use the positive torus action (see Remark 2.2) to define  $V' := \{(\frac{x_1}{c_1}, \dots, \frac{x_n}{c_n}) : x \in V\} \in S_D$ , so that  $(c_1 v_1, \dots, c_n v_n) \in V'^\perp \cap W$ . Since  $\dim(V'^\perp \cap W) = 1$ , we get  $\text{span}(w) = V'^\perp \cap W \in \mathcal{B}_D(W)$ . ■

**Remark 8.3.** If  $M$  is the positroid corresponding to a J-diagram  $D$ , with dual positroid  $M^*$ , then by Lemma 3.3(ii) we have  $\mathcal{V}(D) = \{\text{alt}(\text{sign}(u)) : u \in U\}$  for any  $U \in S_{M^*}$ . Therefore by Lemma 3.3(i), determining which sign vectors in  $\mathcal{V}(D)$  minimize  $\overline{\text{var}}(\cdot)$  is equivalent to determining which sign vectors in  $\text{sign}(U)$  (for any  $U \in S_{M^*}$ ) maximize  $\text{var}(\cdot)$ .

Recall that  $\overline{\mathcal{D}_{n,k,1}}$  is the set of J-diagrams with at most one  $+$  per row, and each  $+$  appears at the right end of its row. We showed in Proposition 5.16 that for  $D \in \overline{\mathcal{D}_{n,k,1}}$ ,  $\mathcal{V}(D)$  contains a unique sign vector  $\sigma$  (up to multiplication by  $\pm 1$ ) with  $\overline{\text{var}}(\sigma) = k$ , which we denoted by  $\sigma(D)$  (Definition 5.7). In this case, we have  $\mathcal{B}_D(W) = \mathcal{B}_{\sigma(D)}(W)$ , as verified by Lemma 8.2. Also note that by Theorem 5.17, the dimension of  $\mathcal{B}_D(W)$  is the number of  $+$ 's in  $D$  (see Figure 11). We now give a formula for  $\dim(\mathcal{B}_D(W))$  for any J-diagram  $D$ .

**Proposition 8.4** (Dimension of the image of an arbitrary cell).

Let  $D$  be a J-diagram of type  $(k, n)$ . Then the dimension of  $\mathcal{B}_D(W)$  is the number of rows of  $D$  which contain a  $+$ .

This implies that a cell of  $\text{Gr}_{k,n}^{\geq 0}$  has its dimension preserved when mapped by  $\tilde{Z}$  to the  $m = 1$  amplituhedron if and only if its  $\mathbb{J}$ -diagram has at most one  $+$  in each row. This is related to the notion of *kinematical support* (see [AHBC<sup>+</sup>16, Chapter 10], [Lam16a, Definition 4.3]).

**Proof.** Label the steps of the southeast border of  $D$  by  $1, \dots, n$  from northeast to southwest, and denote by  $I \subseteq [n]$  the set of  $i \in [n]$  such that  $i$  labels a vertical step whose row contains no  $+$ 's. We will show that the codimension of  $\mathcal{B}_D(W)$  equals  $|I|$ .

It follows from Definition 2.13 that  $I$  is the set of coloops of  $M(D)$ , i.e.  $I = \{i \in [n] : e^{(i)} \in V\}$  for any  $V \in S_D$ , where  $e^{(i)}$  denotes the  $i$ th unit vector. Hence  $I = \{i \in [n] : \sigma_i = 0 \text{ for all } \sigma \in \mathcal{V}(D)\}$ . Now let  $\mathcal{H}^W$  be a hyperplane arrangement from Definition 6.10, so that  $B(\mathcal{H}^W)$  is homeomorphic to  $\mathcal{B}_{n,k,1}(W)$  by Theorem 6.16. By Lemma 8.2 we have

$$\mathcal{B}_D(W) \subseteq \bigcup \{\mathcal{B}_\sigma(W) : \sigma \in \overline{\mathbb{P}\text{Sign}_{n,k,1}} \text{ with } \sigma_i = 0 \text{ for all } i \in I\},$$

so the image of  $\mathcal{B}_D(W)$  in  $B(\mathcal{H}^W)$  under the homeomorphism  $\mathcal{B}_{n,k,1}(W) \rightarrow B(\mathcal{H}^W)$  is contained in  $\bigcap_{i \in I} H_i$  by Theorem 6.16. Since  $\mathcal{H}^W$  is generic (Definition 6.4), the codimension of  $\bigcap_{i \in I} H_i$  equals  $|I|$ , so the codimension of  $\mathcal{B}_D(W)$  is at least  $|I|$ .

Conversely, note that any  $X \in \text{Gr}_{l,n}$  contains a vector which changes sign at least  $l - 1$  times: put an  $l \times n$  matrix whose rows span  $X$  into reduced row echelon form, and take the alternating sum of the rows. Now fix any  $V \in S_D$ . The element  $\text{alt}(V^\perp) \in \text{Gr}_{n-k,n}$  contains a vector  $\text{alt}(v)$  (for some  $v \in V^\perp$ ) which changes sign at least  $n - k - 1$  times. Since for all  $i \in [n] \setminus I$  there exists  $w \in V^\perp$  with  $w_i \neq 0$ , we may perturb  $v \in V^\perp$  to make the components  $[n] \setminus I$  all nonzero, without changing the sign of any nonzero components of  $v$ . We obtain a vector  $v' \in V^\perp$  satisfying  $\text{var}(\text{alt}(v')) \geq n - k - 1$  and  $v'_i \neq 0$  for all  $i \in [n] \setminus I$ . By Lemma 3.3(i), we have  $\overline{\text{var}}(v') \leq k$ , and since  $\overline{\text{var}}(v') \geq k$  by Theorem 3.4(i), we get  $\overline{\text{var}}(v') = k$ . Letting  $\sigma := \text{sign}(v')$ , we have  $\mathcal{B}_\sigma(W) \subseteq \mathcal{B}_D(W)$  by Lemma 8.2. The codimension of  $\mathcal{B}_\sigma(W)$  equals the number of zero components of  $\sigma$ , which is at most  $|I|$ . Hence the codimension of  $\mathcal{B}_D(W)$  is at most  $|I|$ .  $\blacksquare$

Next we determine which cells  $S_D$  are mapped injectively to the  $m = 1$  amplituhedron, and explicitly describe the images of such cells. In order to do so, we introduce a new set of  $\mathbb{J}$ -diagrams, and define a ‘sliding’ operation on the  $+$ 's of such diagrams.

**Definition 8.5.** Let  $\overline{\mathcal{L}_{n,k,1}}$  denote the set of  $\mathbb{J}$ -diagrams of type  $(k, n)$  which have at most one  $+$  in each row, and which satisfy the *L-condition*: there is no 0 which has a  $+$  above it in the same column and a  $+$  to the right in the same row. (However, we do not require each  $+$  to appear at the right end of its row.) We let  $\mathcal{L}_{n,k,1}$  denote the subset of  $\overline{\mathcal{L}_{n,k,1}}$  of  $\mathbb{J}$ -diagrams with exactly  $k$   $+$ 's.

Note that  $\mathcal{D}_{n,k,1} \subseteq \mathcal{L}_{n,k,1}$  and  $\overline{\mathcal{D}_{n,k,1}} \subseteq \overline{\mathcal{L}_{n,k,1}}$ . For example, we have

$$\mathcal{D}_{2,2,1} = \left\{ \begin{array}{|c|c|} \hline 0 & + \\ \hline 0 & + \\ \hline \end{array}, \begin{array}{|c|c|} \hline 0 & + \\ \hline + & \\ \hline \end{array}, \begin{array}{|c|} \hline + \\ \hline + \\ \hline \end{array} \right\} \quad \text{and} \quad \mathcal{L}_{2,2,1} = \mathcal{D}_{2,2,1} \sqcup \left\{ \begin{array}{|c|c|} \hline + & 0 \\ \hline + & 0 \\ \hline \end{array}, \begin{array}{|c|c|} \hline + & 0 \\ \hline + & \\ \hline \end{array} \right\}.$$

**Remark 8.6.** It is not hard to see that  $\mathcal{L}_{n,k,1}$  (respectively,  $\overline{\mathcal{L}_{n,k,1}}$ ) consists of the  $\mathbb{J}$ -diagrams we obtain from diagrams in  $\mathcal{D}_{n',k,1}$  (respectively,  $\overline{\mathcal{D}_{n',k,1}}$ ) by inserting  $n - n'$  columns of all 0's, as  $n'$  ranges over all  $n' \leq n$ .



**Definition 8.7.** Given a  $\mathcal{J}$ -diagram  $D \in \overline{\mathcal{L}_{n,k,1}}$ , we denote by  $\text{Slide}(D)$  the set of  $\mathcal{J}$ -diagrams which can be obtained from  $D$  by doing the following for each  $+$  of  $D$ .

- (1) Slide the  $+$  weakly to the right somewhere in the same row, say from box  $b$  to box  $b'$ , such that the southeast corner of box  $b'$  lies on the southeast border of  $D$ .
- (2) Remove all boxes to the right of box  $b'$  in the same row.
- (3) If  $b' \neq b$  and the entire lower edge of  $b'$  lies on the southeast border of  $D$ , we can choose to remove box  $b'$  (or not).

Note that  $\text{Slide}(D) \subseteq \overline{\mathcal{D}_{n,k,1}}$ .

**Example 8.8.** Let

$$D := \begin{array}{|c|c|c|c|} \hline 0 & + & 0 & 0 \\ \hline 0 & 0 & 0 & \\ \hline 0 & + & 0 & \\ \hline 0 & + & 0 & \\ \hline + & & & \\ \hline \end{array} \in \overline{\mathcal{L}_{5,9,1}}.$$

Then  $\text{Slide}(D)$  equals the set of  $\mathcal{J}$ -diagrams appearing below.

$$\begin{array}{|c|c|c|} \hline 0 & 0 & + \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & + \\ \hline 0 & + & \\ \hline + & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 0 & 0 & + \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & + \\ \hline 0 & 0 & + \\ \hline + & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 0 & 0 & + \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & + \\ \hline 0 & 0 & \\ \hline + & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & + \\ \hline 0 & 0 & 0 & \\ \hline 0 & 0 & + & \\ \hline 0 & + & & \\ \hline + & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & + \\ \hline 0 & + & \\ \hline + & & \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & + \\ \hline 0 & 0 & 0 & \\ \hline 0 & 0 & + & \\ \hline 0 & 0 & + & \\ \hline + & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & + \\ \hline 0 & 0 & 0 & \\ \hline 0 & 0 & + & \\ \hline 0 & 0 & & \\ \hline + & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & + \\ \hline 0 & 0 & + \\ \hline + & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & + \\ \hline 0 & 0 & \\ \hline + & & \\ \hline \end{array}$$

◇

**Example 8.9.** If  $D \in \overline{\mathcal{D}_{n,k,1}}$ , then each  $+$  of  $D$  is in the rightmost box of its row, so we cannot slide it further to the right. Hence  $\text{Slide}(D) = \{D\}$ . ◇

**Theorem 8.10.** Let  $D$  be a  $\mathcal{J}$ -diagram of type  $(k, n)$ . Then the map  $S_D \rightarrow \mathcal{B}_{n,k,1}(W)$ ,  $V \mapsto V^\perp \cap W$  is injective on the cell  $S_D$  if and only if  $D \in \overline{\mathcal{L}_{n,k,1}}$ . In this case, we have

$$(8.11) \quad \mathcal{B}_D(W) = \bigsqcup_{D' \in \text{Slide}(D)} \mathcal{B}_{D'}(W).$$

We will prove Theorem 8.10 over the remainder of the section, divided into several steps. First, we consider two examples.

**Example 8.12.** Let  $D$  be the  $\mathcal{J}$ -diagram from Example 8.8. Then Theorem 8.10 asserts that  $S_D$  maps injectively to the  $m = 1$  amplituhedron, and that its image is the disjoint union of the images of the 9 cells corresponding to the  $\mathcal{J}$ -diagrams in  $\text{Slide}(D)$ . ◇

**Example 8.13.** Theorem 8.10 implies that if two cells of  $\text{Gr}_{k,n}^{\geq 0}$  map injectively to the  $m = 1$  amplituhedron, then their images are distinct. This can fail to hold if we do not assume that the cells map injectively. For example, if  $n = 3$  and  $k = 1$ , then the cells  $\{(1 : 0 : a) : a > 0\}$  and  $\{(1 : b : c) : b, c > 0\}$  of  $\text{Gr}_{1,3}^{\geq 0}$  have the same image by Lemma 8.2, namely  $\mathcal{B}_{(+,-,-)}(W) \sqcup \mathcal{B}_{(+,0,-)}(W) \sqcup \mathcal{B}_{(+,+, -)}(W)$ .  $\diamond$

Now we begin proving Theorem 8.10. Let us characterize the  $\mathbb{J}$ -diagrams in  $\overline{\mathcal{L}_{n,k,1}}$  by a matroid-theoretic condition.

**Definition 8.14.** Let  $D$  be a  $\mathbb{J}$ -diagram of type  $(k, n)$ . The set of sign vectors  $\mathcal{V}(D)$  from Definition 8.1 is an induced subposet of  $\{0, +, -\}^n$  from Definition 5.3. We call the minimal elements of  $\mathcal{V}(D) \setminus \{0\}$  the *circuits*<sup>8</sup> of  $D$ , and denote the set of circuits by  $\mathcal{C}(D)$ . A more concrete way to think about circuits is the following. Fix  $V \in S_D$  and an  $(n-k) \times n$  matrix  $A$  whose rows span  $V^\perp$ . For a subset  $I \in \binom{[n]}{n-k}$  such that columns  $I$  of  $A$  form a basis for  $\mathbb{R}^{n-k}$ , let  $A(I)$  be the  $(n-k) \times n$  matrix whose rows span  $V^\perp$  which we obtain by row reducing  $A$  so as to get an identity matrix in columns  $I$ . Then the sign vectors of rows appearing in some matrix  $A(I)$  are precisely the circuits of  $D$  (up to sign).

We call  $i \in [n]$  a *loop* of  $D$  if the unit vector  $e^{(i)}$  is contained in  $\mathcal{V}(D)$ , and a *coloop* if  $\sigma_i = 0$  for all  $\sigma \in \mathcal{V}(D)$ . These are precisely the loops and coloops (from Definition 2.10) of the positroid  $M(D)$  from Definition 2.13. Alternatively, loops label the columns of  $D$  which contain only 0's, and coloops label the rows which contain only 0's.

**Lemma 8.15.** *Let  $D$  be a  $\mathbb{J}$ -diagram of type  $(k, n)$ . Then  $D \in \overline{\mathcal{L}_{n,k,1}}$  if and only if there do not exist  $\sigma \in \mathcal{C}(D)$  and  $a < b < c$  in  $[n]$  such that*

- (i)  $\sigma_a, \sigma_c \neq 0$ ;
- (ii)  $\sigma_b = 0$ ; and
- (iii)  $b$  is neither a loop nor a coloop of  $D$ .

**Proof.** Let  $D'$  be the  $\mathbb{J}$ -diagram contained in a  $k' \times (n' - k')$  rectangle obtained from  $D$  by deleting all rows and columns which contain only 0's. Equivalently, the positroid  $M(D')$  is obtained from  $M(D)$  by restricting the ground set  $[n]$  (as in Definition 2.18) to elements which are neither loops nor coloops. Then we obtain  $\mathcal{C}(D')$  from  $\mathcal{C}(D)$  by deleting the circuits  $\pm e^{(i)}$  corresponding to loops  $i$ , restricting all circuits from  $[n]$  to  $\{i \in [n] : i \text{ is neither a loop nor a coloop of } D\}$ , and multiplying certain components by  $-1$ . Note that  $D'$  has no loops or coloops, and  $D \in \overline{\mathcal{L}_{n,k,1}}$  if and only if  $D' \in \mathcal{D}_{n',k',1}$ . Hence it suffices to prove that  $D' \in \mathcal{D}_{n',k',1}$  if and only if there do not exist  $\sigma \in \mathcal{C}(D')$  and  $a < b < c$  in  $[n]$  such that  $\sigma_a, \sigma_c \neq 0$  and  $\sigma_b = 0$ .

( $\Rightarrow$ ): Suppose that  $D' \in \mathcal{D}_{n',k',1}$ . Let us fix  $V' \in S_{D'}$ , so that the representable matroid  $M(V'^\perp)$  (defined in Example 2.11) equals  $M(D')^*$ . By Lemma 5.10,  $M(D')^*$  is the direct sum of uniform matroids of rank 1 whose ground sets are all intervals. That is, we can write

<sup>8</sup>In the language of oriented matroids,  $\mathcal{C}(D)$  is the set of (signed) circuits of the oriented matroid represented by any  $V \in S_D$ . The supports of circuits are precisely the (unsigned) circuits of the (unoriented) matroid  $M(V)$ . What we call a circuit will always be a sign vector.

$V'^\perp$  as the row span of an  $(n' - k') \times n'$  matrix of the form

$$(8.16) \quad \begin{bmatrix} * & \cdots & * & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots \\ 0 & \cdots & 0 & * & \cdots & * & 0 & \cdots & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & * & \cdots & * & \cdots \\ \vdots & & & \vdots & & & \vdots & & & \ddots \end{bmatrix},$$

where all the  $*$ 's are nonzero. The sign vectors of these rows, and their negations, are precisely the circuits of  $D'$ . We see that the circuits of  $D'$  satisfy the required condition.

( $\Leftarrow$ ): Suppose that there do not exist  $\sigma \in \mathcal{C}(D')$  and  $a < b < c$  in  $[n]$  such that  $\sigma_a, \sigma_c \neq 0$  and  $\sigma_b = 0$ . Let us again take some  $V' \in S_{D'}$ , and put an  $(n' - k') \times n'$  matrix whose rows span  $V'^\perp$  into reduced row echelon form. Then the sign vector of each row of this matrix is a circuit of  $D'$ , and so the nonzero entries in each row form a consecutive block. It follows that this matrix is of the form (8.16), where all the  $*$ 's are nonzero. (The matrix has no zero columns since  $D'$  has no coloops.) Hence the matroid  $M(D')^* = M(V'^\perp)$  is the direct sum of uniform matroids of rank 1 whose ground sets are all intervals, whence  $D' \in \mathcal{D}_{n',k',1}$  by Lemma 5.10.  $\blacksquare$

**Lemma 8.17.** *Let  $D$  be a  $\mathcal{J}$ -diagram of type  $(k, n)$ . Suppose that there exist a circuit  $\sigma \in \mathcal{C}(D)$  and  $a < b < c$  in  $[n]$  such that*

- (i)  $\sigma_a, \sigma_c \neq 0$ ;
- (ii)  $\sigma_b = 0$ ; and
- (iii)  $b$  is not a coloop of  $D$ .

*Then there exists  $\tau \in \mathcal{V}(D)$  with  $\overline{\text{var}}(\tau) = k$  and  $\tau_b = 0$ .*

We will use such a sign vector  $\tau$  in Corollary 8.18 to provide a certificate of non-injectivity, if  $D \notin \overline{\mathcal{L}_{n,k,1}}$ . In proving Corollary 8.18, we will want to require that  $b$  (satisfying  $\tau_b = 0$ ) is not a loop of  $D$ , as in Lemma 8.15. But in order to prove just Lemma 8.17, we do not need to assume that  $b$  is not a loop.

**Proof.** Fix  $V \in S_D$  and an  $(n - k) \times n$  matrix  $A$  whose rows span  $U := \text{alt}(V^\perp)$ . By Lemma 3.3(i), it suffices to show that there exists  $w \in U$  with  $\text{var}(w) = n - k - 1$  and  $w_b = 0$ . We will use the fact that  $U \in \text{Gr}_{n-k,n}^{\geq 0}$ , by Lemma 3.3(ii). The idea is to take  $u \in U$  with sign vector  $\sigma' := \text{alt}(\sigma)$ , and perturb it so that it has  $n - k - 1$  sign changes. The presence of both  $a$  and  $c$  will allow us to get a ‘free’ sign change without ‘using’  $b$ .

Let  $J := \{j \in [n] : \sigma_j = 0\}$ . The fact that  $\sigma$  is a circuit implies that columns  $J$  of  $A$  do not span  $\mathbb{R}^{n-k}$ , but columns  $J \cup \{a\}$  do span  $\mathbb{R}^{n-k}$ . Also, since  $b$  is not a coloop of  $D$ , column  $b$  of  $A$  is nonzero. Hence we can extend  $\{a, b\}$  to  $I \in \binom{J \cup \{a\}}{n-k}$  such that columns  $I$  of  $A$  form a basis of  $\mathbb{R}^{n-k}$ . Let  $A(I)$  denote the  $(n - k) \times n$  matrix whose rows span  $U$  which we obtain by row reducing  $A$  so as to get an identity matrix in columns  $I$ . We denote by  $v^{(i)}$  the row of  $A(I)$  whose pivot column is  $i$ , i.e.  $v_i^{(i)} = 1$  and  $v_j^{(i)} = 0$  for all  $j \in I \setminus \{i\}$ .

*Claim.*  $\sigma'_c = (-1)^{|I \cap (a,c)|} \sigma'_a$ , where  $(a, c)$  denotes the open interval between  $a$  and  $c$ .

*Proof of Claim.* Write  $I = \{i_1 < \dots < i_{n-k}\}$ , so that  $a = i_r$  for some  $r \in [n-k]$ . Since  $u_{i_s} = 0$  for all  $s \in [n-k] \setminus \{r\}$ , we can use  $v^{(i_s)}$  to perturb  $u$ , without changing the sign of  $u_c$ , so that component  $i_s$  gets the sign

$$\begin{cases} (-1)^{s-r} \sigma'_a, & \text{if } i_s < c \\ (-1)^{s-r+1} \sigma'_a, & \text{if } i_s > c \end{cases}.$$

Since the resulting vector  $u'$  lies in  $U \in \text{Gr}_{n-k,n}^{\geq 0}$ , we get  $\text{var}(u') \leq n-k-1$  by Theorem 3.4(i). Hence  $u'$  does not alternate in sign on  $I \cup \{c\}$ , and so  $\sigma'_c \neq -(-1)^{|I \cap (a,c)|} \sigma'_a$ .  $\square$

We obtain our desired vector  $w \in U$  as follows. Write  $I \setminus \{b\} = \{i'_1 < \dots < i'_{n-k-1}\}$ , so that  $a = i'_r$  for some  $r \in [n-k]$ . Then for  $s \in [n-k-1] \setminus \{r\}$ , we use  $v^{(i'_s)}$  to perturb  $u$ , without changing the sign of  $u_c$ , so that component  $i'_s$  gets sign

$$\begin{cases} (-1)^{s-r} \sigma'_a, & \text{if } i'_s < c \\ (-1)^{s-r+1} \sigma'_a, & \text{if } i'_s > c \end{cases}.$$

The resulting vector  $w$  alternates in sign on  $(I \setminus \{b\}) \cup \{c\}$  by the claim, so  $\text{var}(w) \geq n-k-1$ . We have  $\text{var}(w) \leq n-k-1$  by Theorem 3.4(i), and  $w_b = 0$  by construction.  $\blacksquare$

**Corollary 8.18.** *Let  $D$  be a  $\mathcal{J}$ -diagram of type  $(k, n)$  which is not in  $\overline{\mathcal{L}_{n,k,1}}$ . Then the map  $S_D \rightarrow \mathcal{B}_{n,k,1}(W)$ ,  $V \mapsto V^\perp \cap W$  is not injective.*

**Proof.** Fix  $V \in S_D$ . By Lemma 8.15 and Lemma 8.17, there exist  $w \in V^\perp$  and  $b \in [n]$  such that  $\overline{\text{var}}(w) = k$ ,  $w_b = 0$ , and  $b$  is neither a loop nor a coloop of  $D$ . The strategy of the proof is to use the positive torus action (see Remark 2.2) so as to get from  $w$  an element of  $\mathcal{B}_{n,k,1}(W)$ , and then use the positive torus action again in component  $b$  to get many elements of  $S_D$  which map to our chosen element of  $\mathcal{B}_{n,k,1}(W)$ .

Let  $\sigma := \text{sign}(w)$ , so that by Lemma 3.7(i) there exists  $w' \in W$  with  $\text{sign}(w') = \sigma$ . Letting  $w = (w_1, \dots, w_n)$ , we can therefore write  $w' = (c_1 w_1, \dots, c_n w_n)$  for some  $c_1, \dots, c_n > 0$ . Letting  $V' := \{(\frac{v_1}{c_1}, \dots, \frac{v_n}{c_n}) : v \in V\} \in S_D$ , we have  $w' \in V'^\perp$ . Then for  $t > 0$ , we define

$$V'_t := \{(v_1, \dots, v_{b-1}, t v_b, v_{b+1}, \dots, v_n) : v \in V'\} \in S_D.$$

*Claim.* The  $V'_t$  ( $t > 0$ ) are all distinct.

*Proof of Claim.* Let  $M(D)$  denote the matroid from Definition 2.13. Since  $b$  is not a loop or a coloop of  $D$ , there exist  $I, J \in M(D)$  with  $b \in I$  and  $b \notin J$ . Then  $\frac{\Delta_I(V'_t)}{\Delta_J(V'_t)} \cdot \frac{\Delta_J(V')}{\Delta_I(V')} = t$  for  $t > 0$ .  $\square$

Since  $w' \in V'^\perp \cap W$  and  $\dim(V'^\perp \cap W) = 1$ , the elements  $V'_t$  ( $t > 0$ ) all map to the line spanned by  $w'$  in  $\mathcal{B}_{n,k,1}(W)$ .  $\blacksquare$

We now show that if  $D \in \overline{\mathcal{L}_{n,k,1}}$ , then  $S_D$  maps injectively to the  $m = 1$  amplituhedron, and we also determine its image. We need to understand the sign vectors  $\mathcal{V}(D)$  for such  $D$ . This can be done, thanks to the fact that when we remove enough loops (columns containing only 0's) from  $D$ , we obtain an element of  $\overline{\mathcal{D}_{n,k,1}}$  (see Remark 8.6). We first consider the case that  $D$  is in  $\mathcal{L}_{n,k,1}$  (i.e.  $D$  has no coloops) and its +’s lie in a single column.

**Lemma 8.19.** *Suppose that  $D \in \mathcal{L}_{n,k,1}$  with  $k \geq 1$  such that the  $+$ 's of  $D$  lie in a single column. Then the map  $S_D \rightarrow \mathcal{B}_{n,k,1}(W)$ ,  $V \mapsto V^\perp \cap W$  is injective on  $S_D$ , and  $\mathcal{B}_D(W) = \bigsqcup_{D' \in \text{Slide}(D)} \mathcal{B}_{D'}(W)$ .*

**Proof.** Let  $L \in \binom{[n]}{n-k-1}$  denote the set of loops of  $D$  (from Definition 8.14). That is, if we label the steps of the southeast border of  $D$  from 1 to  $n$ , then  $L$  is the set of labels of the horizontal steps, excluding the label corresponding to the column of  $+$ 's. Then  $M(D)$  is a matroid of rank  $k$ , which is uniform when restricted to  $[n] \setminus L$  and has set of loops  $L$ . It follows from Lemma 3.7(ii) that  $\sigma \in \mathcal{V}(D)$  if and only if

$$\sigma|_{[n] \setminus L} \in \{0, (+, -, +, -, \dots), (-, +, -, +, \dots)\}.$$

(The sign of  $\sigma_i$  for  $i \in L$  is arbitrary.)

To compute  $\mathcal{B}_D(W)$ , we use Lemma 8.2. We need to identify those  $\sigma \in \mathcal{V}(D)$  satisfying  $\overline{\text{var}}(\sigma) = k$ . Let us first consider the example  $D = \begin{bmatrix} + & 0 & 0 \end{bmatrix}$ , so  $n = 4$ ,  $k = 1$ ,  $L = \{2, 3\}$ . We have

$$\mathcal{V}(D) = \{(+, *, *, -)\} \cup \{(-, *, *, +)\} \cup \{(0, *, *, 0)\},$$

where each  $*$  can be 0,  $+$ , or  $-$ . Since  $\overline{\text{var}}(0, *, *, 0) \geq 2$ , the sign vectors  $\sigma \in \mathcal{V}(D)$  with  $\overline{\text{var}}(\sigma) = 1$  (modulo multiplication by  $\pm 1$ ) are

$$(+, +, +, -), (+, +, -, -), (+, -, -, -), (+, +, 0, -), (+, 0, -, -),$$

that is, a sequence of  $+$ 's followed by  $-$ 's, possibly with one 0 in between. The corresponding J-diagrams (from Definition 5.7) are

$$\begin{bmatrix} + \end{bmatrix}, \quad \begin{bmatrix} 0 & + \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & + \end{bmatrix}, \quad \begin{bmatrix} 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \end{bmatrix},$$

respectively.

Now we describe what happens in general. Write  $[n] \setminus L = \{r_1 < \dots < r_{k+1}\}$ . Given  $\sigma \in \{0, +, -\}^n$ , we have  $\sigma \in \mathcal{V}(D)$  and  $\overline{\text{var}}(\sigma) = k$  if and only if

- $\sigma$  alternates in sign on  $[n] \setminus L$ ;
- $\sigma_i = \sigma_{r_1}$  for all  $i < r_1$ ;
- $\sigma_i = \sigma_{r_{k+1}}$  for all  $i > r_{k+1}$ ; and
- for all  $j \in [k]$ , there exists an integer  $s_j \in [r_j, r_{j+1} - 1]$  such that  $\sigma_i = \sigma_{r_j}$  for all  $i \in [r_j, s_j]$ ,  $\sigma_i = \sigma_{r_{j+1}}$  for all  $i \in [s_j + 2, r_{j+1}]$ , and if  $s_j + 1 \neq r_{j+1}$  then  $\sigma_{s_j+1}$  equals either 0 or  $\sigma_{r_{j+1}}$ .

We obtain the J-diagram  $\Omega_{DS}^{-1}(\sigma)$  (see Definition 5.7) by a slide as in Definition 8.7, as follows. In step (1), we slide the  $+$  in the  $j$ th row of  $D$  so that it is  $s_j - r_j$  boxes to the left of the rightmost box in the row. In step (2), we remove the boxes to the right of this  $+$ . In step (3), we remove the box  $b'$  containing this  $+$  if and only if  $\sigma_{s_j+1} = 0$ . (Note that the case when  $b' \neq b$  and the bottom edge of  $b'$  lies on the southeast border of  $D$  corresponds to  $s_j + 1 \neq r_{j+1}$ .) Thus  $\Omega_{DS}^{-1}$  restricted to  $\{\sigma \in \mathcal{V}(D) : \overline{\text{var}}(\sigma) = k\}$  is a bijection  $\{\sigma \in \mathcal{V}(D) : \overline{\text{var}}(\sigma) = k\} \rightarrow \text{Slide}(D)$ , with inverse  $\Omega_{DS}$ . Then Lemma 8.2 implies (8.11).

To see that the map  $S_D \rightarrow \mathcal{B}_{n,k,1}(W)$  is injective, suppose that  $w \in W$  with  $\text{sign}(w) \in \mathcal{V}(D)$  and  $\overline{\text{var}}(w) = k$ . We have just observed that  $w$  is nonzero when restricted to  $[n] \setminus L$ , so the vectors  $w$  and  $e^{(i)}$  for  $i \in L$  are linearly independent. Their span is an element of  $\text{Gr}_{n-k,n}$ , whose orthogonal complement is the unique  $V \in S_D$  with  $V^\perp \cap W = \text{span}(w)$ . ■

**Proof** (of Theorem 8.10). If  $D \notin \overline{\mathcal{L}_{n,k,1}}$ , then the map  $S_D \rightarrow \mathcal{B}_{n,k,1}(W)$  is not injective by Corollary 8.18. Now suppose that  $D \in \overline{\mathcal{L}_{n,k,1}}$ . We must show that the map  $S_D \rightarrow \mathcal{B}_{n,k,1}(W)$  is injective, and that its image equals  $\bigsqcup_{D' \in \text{Slide}(D)} \mathcal{B}_{D'}(W)$ . We first show how to reduce to the case that  $D \in \mathcal{L}_{n,k,1}$ , i.e.  $D$  has no coloops (see Definition 8.14). Let us suppose that  $i \in [n]$  is a coloop of  $D$ , and explain what happens when we delete  $i$ . In terms of  $\mathbb{J}$ -diagrams,  $i$  labels a row of  $D$  which contains only 0's, and deleting this row gives a  $\mathbb{J}$ -diagram  $D'$  indexing a cell of  $\text{Gr}_{k-1,n-1}^{\geq 0}$ . In terms of cells, we have the map

$$\begin{aligned} \{V^\perp : V \in S_D\} &\rightarrow \{V'^\perp : V' \in S_{D'}\}, \\ \{(v_1, \dots, v_{i-1}, 0, v_{i+1}, \dots, v_n)\} &\mapsto \{(v_1, \dots, v_{i-1}, -v_{i+1}, \dots, -v_n)\}. \end{aligned}$$

In terms of the subspace  $W \in \text{Gr}_{k+1,n}^{\geq 0}$ , we map it to

$$W' := \{(w_1, \dots, w_{i-1}, -w_{i+1}, \dots, -w_n) : w \in W \text{ with } w_i = 0\},$$

where  $W' \in \text{Gr}_{k,n-1}^{\geq 0}$  by Theorem 3.4(ii). Note that the map  $V \mapsto V^\perp \cap W$  is injective on  $S_D$  if and only if the map  $V' \mapsto V'^\perp \cap W'$  is injective on  $S_{D'}$ . We also have a bijection  $\text{Slide}(D) \rightarrow \text{Slide}(D')$  given by deleting the row labeled by  $i$ . By Lemma 8.2, this shows that the result for  $D'$  implies the result for  $D$ . Hence we may assume that  $D$  has no coloops, i.e.  $D \in \mathcal{L}_{n,k,1}$ .

In this case, we can decompose  $D$  as follows: there exist  $\mathbb{J}$ -diagrams  $D_1, \dots, D_l$ , where each  $D_j$  has no coloops and all its '+'s lie in a single column, such that we get  $D$  by gluing together  $D_1, \dots, D_l$  from northeast to southwest (so that the southwest corner of  $D_j$  and the northeast corner of  $D_{j+1}$  coincide, for all  $j \in [l-1]$ ), and filling the space above and to the left

of the resulting diagram with 0's. For example, if  $D = \begin{array}{|c|c|c|} \hline 0 & 0 & + \\ \hline 0 & 0 & + \\ \hline + & & \\ \hline \end{array}$ , then  $D_1 = \begin{array}{|c|c|} \hline 0 & + \\ \hline 0 & + \\ \hline \end{array}$  and

$D_2 = \begin{array}{|c|} \hline + \\ \hline \end{array}$ . We have  $M(D) = M(D_1) \oplus \dots \oplus M(D_l)$ , so any  $V \in S_D$  can be written uniquely as  $V_1 \oplus \dots \oplus V_l$ , where  $V_j \in S_{D_j}$ . Now for  $j \in [l]$ , let  $E_j$  be the ground set of  $M(D_j)$ , and let  $U_j$  denote the orthogonal complement of  $V_j$  in  $\mathbb{R}^{E_j}$ , so that  $V^\perp = U_1 \oplus \dots \oplus U_l$ . It follows that the vectors  $w \in V^\perp$  satisfying  $\overline{\text{var}}(w) = k$  can be written precisely as  $w = u_1 + \dots + u_l$ , where  $u_j \in U_j$  with  $\overline{\text{var}}(u_j) = \dim(V_j)$  for all  $j \in [l]$ , and the signs of the  $u_j$ 's are such that we never get an extra sign change from  $u_j$  to  $u_{j+1}$  for  $j \in [l-1]$ . Thus it suffices to verify injectivity and (8.11) locally for each  $D_j$ , which we can do thanks to Lemma 8.19.  $\blacksquare$

## 9. RELAXING TO GRASSMANN POLYTOPES

In this section we discuss what happens when we relax the condition that  $W \in \text{Gr}_{k+m,n}$  is totally positive, in the sense of Lam's *Grassmann polytopes* [Lam16b].

**Definition 9.1** ([Lam16b, Definition 15.1]). Let  $Z$  be a real  $r \times n$  matrix with row span  $W \in \text{Gr}_{k+m,n}$ , where  $k+m \leq r$  (we allow  $Z$  to not have full row rank). Suppose that  $W$  contains a totally positive  $k$ -dimensional subspace. Then by [Lam16b, Proposition 15.2], the map  $\tilde{Z} : \text{Gr}_{k,n}^{\geq 0} \rightarrow \text{Gr}_{k,r}$  is well defined, i.e.  $\dim(Z(V)) = k$  for all  $V \in \text{Gr}_{k,n}^{\geq 0}$  (see Remark 3.17). We call the image  $\tilde{Z}(\overline{S_M}) \subseteq \text{Gr}_{k,r}$  of the closure of a cell  $S_M$  of  $\text{Gr}_{k,n}^{\geq 0}$  a

*Grassmann polytope.* When  $\overline{S}_M = \text{Gr}_{k,n}^{\geq 0}$ , i.e.  $M$  is the uniform matroid of rank  $k$ , we call the image  $\tilde{Z}(\text{Gr}_{k,n}^{\geq 0})$  a *full Grassmann polytope*.

Analogously, given a cell  $S_M$  of  $\text{Gr}_{k,n}^{\geq 0}$ , let us define a *Grassmann arrangement* as

$$\{V^\perp \cap W : V \in \overline{S}_M\} \subseteq \text{Gr}_m(W).$$

We call this a *full Grassmann arrangement* in the case that  $\overline{S}_M = \text{Gr}_{k,n}^{\geq 0}$ . A Grassmann arrangement is well defined, and homeomorphic to the corresponding Grassmann polytope, by the same arguments which appear in Section 3.2.

The amplituhedron  $\mathcal{A}_{n,k,1}(Z)$  is an example of a full Grassmann polytope (see Lemma 6.9). In the case  $k = 1$ , Grassmann polytopes are precisely polytopes in the projective space  $\text{Gr}_{1,r} = \mathbb{P}^{r-1}$ . Therefore, Grassmann polytopes are a generalization of polytopes into Grassmannians.

Lemma 3.15 generalizes to any full Grassmann arrangement; the proof is the same. In particular, in the case that  $m = 1$ , the analogue of Corollary 3.16 holds.

**Proposition 9.2.** *For  $m = 1$ , the full Grassmann arrangement can be described as follows:*

$$\{V^\perp \cap W : V \in \text{Gr}_{k,n}^{\geq 0}\} = \{w \in \mathbb{P}(W) : \overline{\text{var}}(w) \geq k\} \subseteq \mathbb{P}(W).$$

As in Section 6, if  $m = 1$  we can define a hyperplane arrangement  $\mathcal{H}^W$  depending on  $W$ , and show that the full Grassmann arrangement above is homeomorphic to  $B(\mathcal{H}^W)$ . In slightly more detail, we take a  $k$ -dimensional totally positive subspace  $W'$  of  $W$  as provided by Definition 9.1, and let  $w^{(1)}, \dots, w^{(k)}$  be its basis. We extend this to a basis  $w^{(0)}, w^{(1)}, \dots, w^{(k)}$  of  $W$ , and define  $\mathcal{H}^W$  as in Definition 6.10 (we ignore the requirement that  $w^{(0)}$  is positively oriented). Then Lemma 6.12 holds for  $\mathcal{H}^W$ , with the same proof. This implies (as in the first paragraph of the proof of Lemma 6.13) that the face labels of  $B(\mathcal{H}^W)$  are precisely the sign vectors  $\sigma \in \{0, +, -\}^n$  such that  $\overline{\text{var}}(\sigma) \geq k$  and  $\psi_{\mathcal{H}^W}^{-1}(\sigma) \neq \emptyset$ . We deduce that  $\Psi_{\mathcal{H}^W}$  restricted to  $B(\mathcal{H}^W)$  is a homeomorphism onto the full Grassmann arrangement of  $W$ , as claimed.

We observe that as  $W$  varies, we do not recover all bounded complexes of hyperplane arrangements in this way, since the condition that  $W \in \text{Gr}_{k+1,n}$  contains a subspace in  $\text{Gr}_{k,n}^{\geq 0}$  is very restrictive. Indeed, if  $\sigma$  labels a face of  $B(\mathcal{H}^W)$  for such  $W$ , then  $\overline{\text{var}}(\sigma) \geq k$ .

**Remark 9.3.** As in Remark 3.17, the map  $\tilde{Z} : \text{Gr}_{k,n}^{\geq 0} \rightarrow \text{Gr}_{k,r}$  is well defined if and only if  $\text{var}(v) \geq k$  for all nonzero  $v \in \ker(Z)$  [Kar16, Theorem 4.2]. Hence we could have defined Grassmann arrangements using this (possibly) more general class of matrices  $Z$ . However, in the construction of  $\mathcal{H}^W$  above for  $m = 1$ , it was essential that  $W$  (the row span of  $Z$ ) should have a totally positive  $k$ -dimensional subspace. Indeed, we can see from the proof of Lemma 6.12 that classifying the labels of bounded and unbounded faces of  $\mathcal{H}^W$  uses the fact that  $\text{span}(w^{(1)}, \dots, w^{(k)})$  is totally positive.

Recall from Remark 3.17 that it is unknown whether there exists a matrix  $Z$  such that  $\tilde{Z} : \text{Gr}_{k,n}^{\geq 0} \rightarrow \text{Gr}_{k,r}$  is well defined, but the row span of  $Z$  does not contain a totally positive  $k$ -dimensional subspace. The construction of  $\mathcal{H}^W$  above gives further motivation for resolving this problem, which is equivalent to Problem 3.14(ii).

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